SS4521G - 20–24 January 2014

SUMMARY OF IMPORTANT POINTS DISCUSSED IN THE LECTURE

The following concepts/theories were covered/reviewed in the context of stock price processes evolving on a two-period binomial tree model in discrete time:

1. Itō's lemma was derived on the basis of Taylor series expansion of a function. All higher-order terms after the second-order turn out to go to zero.

   In particular, this lemma states the following: Suppose X_t is a stochastic process whose dynamics is given by the SDE $dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t$. If $f \in C^{2,1}$, then $G_t := f(X_t, t)$ is also a stochastic process with dynamics given by

   $$dG_t = \left(\frac{\partial f}{\partial t} + \mu(X_t, t)\frac{\partial f}{\partial x} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma(X_t, t)^2 \right) dt + \frac{\partial f}{\partial x}\sigma(X_t, t)dW_t.$$

   We spent a considerable amount of time discussing some examples in the lecture to illustrate the above formula within the context of financial pricing. You should review these examples.

2. PDE approach to pricing derivative security: Let $f(S_t, t)$ be a price of a derivative whose underlying follows the dynamics of a geometric Brownian motion (i.e., $dS_t = \mu S_t dt + \sigma S_t dW_t$). We consider a riskless portfolio that enables us to find the value of the derivative. In particular, the holder of the portfolio is:
   • short one derivative security and
   • long an amount of $\frac{\partial f}{\partial S}$ number of shares.

With the aid of this riskless portfolio, we showed that f satisfies the partial differential equation (PDE)

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + rS\frac{\partial f}{\partial S} = rf.$$

Note that μ vanishes and only r and σ^2 appear in the pricing PDE. If the terminal/boundary condition is $f = (K - S_T)^+$ then f gives the price of a European put option.

3. Feyman-Kac's Results (presented in class without proof): If, for instance, f is a solution to the boundary value problem

$$\frac{\partial f}{\partial t}(z,t) + r(z,t)\frac{\partial f}{\partial z} + \frac{1}{2}\sigma^2(z,t)\frac{\partial^2 f}{\partial z^2}(z,t) = r(z,t)f(z,t), \quad f(z,T) = \Phi(z),$$

then f has the probabilistic solution $f(z,t) = E\left[e^{-r(T-t)}\Phi(Z_T)|\mathcal{F}_t\right]$ with $r(z,t) = r$.

4. Change of measure in risk-neutral pricing: One can change measure from P to Q such that
$P: \quad dS_t = \mu S_t dt + \sigma S_t dW_t$
$Q: \quad dS_t = rS_t dt + \sigma S_t dW_t^Q$,
where W_t^Q is a $Q-$Brownian motion and $dW_t^Q = dW_t + \gamma_t dt$. In the Black-Scholes model, $\gamma_t = \dfrac{\mu - r}{\sigma}$.

This is justified by the Cameron-Girsanov-Martin Theorem, which will be discussed in the succeeding lectures.

5. We derived the Black-Scholes-Merton option pricing formula. Starting with $dS_t = \mu S_t dt + \sigma dW_t$ under the measure P, we have to change measure such that under the new measure Q (called risk-neutral measure),

$$dS_t = rS_t dt + \sigma S_t dW_t^Q. \tag{1}$$

Note that from the Feynman-Kac's theorem, the PDE satisfied by the derivative price does not involved μ but only r. This supports why we have the dynamics of the underlying variable in (1).

We wish to find the price $f(S_t, t)$ of a European call option. This price is given by the conditional expected value, under the measure Q, of the discounted pay-off, which is $\max(S_T - X, 0)$. Here, X denotes the strike price in the option contract. All expectations that follow below are assumed to be taken under Q. Assuming $r_u = r$,

$$\begin{aligned} f(S_t, t) &= E^Q\left[\exp\left(-\int_t^T r_u du\right) c(S_T, T)\middle| \mathcal{F}_t\right] \\ &= E_t\left[e^{-r(T-t)}\max(S_T - X, 0)\right] \\ &= e^{-r(T-t)}E_t\left[(S_T - X)I_{\{S_T \geq X\}}\right] \\ &= e^{-r(T-t)}E_t\left[S_T I_{\{S_T \geq X\}} - X I_{\{S_T \geq X\}}\right]. \end{aligned} \tag{2}$$

Note that

$$S_T = S_t \exp\left[\left(r - \frac{\sigma^2}{2}\right)(T - t) + \sigma(W_T - W_t)\right].$$

One may verify this by applying Itô's lemma to obtain (1).

So, if $Y := \left(r - \frac{\sigma^2}{2}\right)(T - t) + \sigma(W_T - W_t)$ then $S_T = S_t \exp Y$ and $Y \sim N\left(\left(r - \frac{\sigma^2}{2}(T - t)\right), \sigma^2(T - t)\right)$.

With the aid of Y, expression in (2) can be simplified. Observe that $S_T \geq K \iff S_t \exp Y \geq X \iff Y \geq \ln\frac{X}{S_t} = -\ln\frac{S_t}{X}$.

Consequently, the first term of the conditional expectation in (2) becomes

$$\begin{aligned} S_t E_t\left[\exp Y I_{\{Y \geq -\ln\frac{S_t}{X}\}}\right] &= S_t \int_{-\ln\frac{S_t}{X}}^{\infty} e^Y f_Y(y) dy \\ &= S_t e^{r(T-t)}\Phi(d_1) \end{aligned} \tag{3}$$

where $d_1 = \dfrac{\ln\frac{S_t}{K} + \left(r + \frac{\sigma^2}{2}\right)(T - t)}{\sigma\sqrt{T - t}}$.

On the other hand, the second term of the conditional expectation in (2) reduces to

$$\begin{aligned} XP(S_T \geq K) &= KP\left(Y \geq -\ln\frac{S_t}{K}\right) = K\int_{-\ln\frac{S_t}{K}}^{\infty} f_Y(y)dy \\ &= K\Phi(d_2) \end{aligned} \tag{4}$$

where $d_2 = \dfrac{\ln\frac{S_t}{K} + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}$.

Plugging-in the results from equations (3) and (4) into equation (2), we have

$$f(S_t, t) = S_t\Phi(d_1) - Xe^{-r(T-t)}\Phi(d_2).$$

6. Equations (3) and (4) are justified by the two lemmas given below whose proofs are part of the required problems for submission in Assignment #1.

   Let $Z \sim N(m,\ \sigma^2)$ and $\Phi(y) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{y} e^{-\frac{s^2}{2}}ds$.

   Lemma 1: $P\left(e^Z > u\right) = \Phi\left(\frac{m-\ln u}{\sigma}\right)$.

   Lemma 2: $E\left[e^Z I_{\{Z>u\}}\right] = e^{m+\frac{\sigma^2}{2}}\Phi\left(\dfrac{m+\sigma^2-u}{\sigma}\right)$ where I is an indicator function.

7. To find the corresponding European put price p_t, we simply use the put-call parity given by the relation $p_t + S_t = c_t + Xe^{-r(T-t)}$, where X is the strike price, c_t is the European call price and S_t is the share price at time t, and $T > t$ is the maturity date. The put-call parity was covered in SS3520.