SS4521G - 24–28 March 2014

SUMMARY OF IMPORTANT POINTS DISCUSSED IN THE LECTURE

The following concepts/theories were covered/reviewed:

1. To find the hedging strategy, we shall look at the PDE satisfied by $v(t, x, y)$. That is, we consider a function $v(t, x, y)$ such that

$$v(t, S(t), Y(t)) = V(t) = E\left[e^{-r(T-t)} \left(\frac{1}{T} Y(T) - X \right)^{+} \middle| \mathcal{F}_t \right], \quad 0 \leq t \leq T.$$

   where E is a risk-neutral expectation, $S(t)$ is the stock price following GBM dynamics and $Y(t) = \displaystyle\int_0^t S(u) du$.

   The function v satisfies the terminal condition $v(T, x, y) = \left(\frac{y}{T} - X \right)^{+}$ for all x and y.

2. We consider the dynamics of the discounted option value $e^{-rt} v(t, S(t), Y(t))$ using Itō's lemma. We note that this discounted option value is a martingale, and therefore the dt component of the dynamics must be identically zero. This gives the PDE satisfied by the European Asian (Eurasian) call option price as

$$\begin{aligned}
&v_t(t, x, y) + r x v_x(t, x, y) + x v_y(t, x, y) + \frac{1}{2} \sigma^2 x^2 v_{xx}(t, x, y) = r v(t, x, y), \\
&0 \leq t < T, \ x > 0, \ y \in \mathbb{R} \\
&v(T, x, y) = \left(\frac{y}{T} - X \right)^{+}, \ x \geq 0, \ y \in \mathbb{R}.
\end{aligned}$$

   Remark: In the derivation of the above PDE, note that $S(t)$ and $Y(t)$ were replaced by the dummy variables x and y, respectively.

The above PDE is similar to the PDE satisfied by the price of a regular European option except for the term $xv_y(t, x, y)$, which is coming from the Y variable (average value).

3. When the dt term of the dynamics of the discounted Asian option value is set to 0, we obtain

$$d(e^{-rt}v(t, S(t), Y(t))) = e^{-rt}\sigma S(t)v_x(t, S(t), Y(t))dW(t). \quad (1)$$

But we note that (the self-financing and replicating portfolio Π for the Asian option value, recall SS 3520B or AM 3613), has dynamics

$$d(e^{-rt}\Pi(t)) = e^{-rt}\sigma S(t)\Delta(t)dW(t), \quad (2)$$

where Δ is the number of shares in the stock investment, and the rest of the replicating portfolio is invested in the bond (or risk-free asset). Thus, by matching equations (1) and (2), we obtain the hedging formula $\Delta(t) = v_x(t, S(t), Y(t))$.

4. It is important to realise that when considering the dt part of the discounted option value, we obtain the **pricing** equation whilst considering the dW_t component gives us the **hedging** formula.

Interest Rate Modelling

5. The stochastic modelling of interest rates is motivated by the pricing and risk management of products with long-term maturities. So far, in the valuation of derivative securities previously discussed, interest rates are treated as constants. This is not a realistic assumption and can have significant impact in the pricing of contingent claims.

6. We consider the term-structure models, which describe the evolution of zero-coupon bonds. We shall focus in the modelling of the short rate

r_t and then look at its implications to zero-coupon bond pricing and yield-curve modelling. The rate r_t is the rate of lending or borrowing applicable to an infinitesimally short period of time at time t.

7. In general, if f_T is the pay-off of a derivative security at time $T > t$, the fair price of the derivative at time t is given by

$$E^Q \left[e^{-\int_t^T r_u du} f_T \middle| \mathcal{F}_t \right],$$

where Q is a risk-neutral measure.

8. Let $B(t,T)$ be the price of a zero-coupon bond at time t for a maturity value of 1 at time T. So, from the risk-neutral valuation formula with $f_T = 1$, we have

$$E^Q \left[e^{-\int_t^T r_u du} 1 \middle| \mathcal{F}_t \right].$$

Typically, r_t is Markov to make the calculation of $B(t,T)$ tractable. Thus,

$$E^Q \left[e^{-\int_t^T r_u du} f_T \middle| r_t = r \right].$$

9. Suppose $Y(t,T)$ is the continuously compounded interest rate at time t for a term $T - t$; $Y(t,T)$ is the yield rate and

$$B(t,T) = e^{-Y(t,T)(T-t)}.$$

Consequently,

$$\begin{aligned} Y(t,T) &= -\frac{1}{T-t} \ln B(t,T) && (3) \\ &= -\frac{1}{T-t} \ln E^Q \left[e^{-\int_t^T r_u(r) du} \right] && (4) \end{aligned}$$

The respective equations in (3) and (4) show that the term structure of interest rates (i.e., evolution of yield rates) can be obtained from the bond price (theoretical or observed) and from the short rate r_t.

Note: Clearly, the modelling of the bond price provides the entire yield curve.

10. As the bond price relies on r_t, we specify a model for r_t, calculate $B(t,T)$ and obtain $Y(t,T)$.

11. There are two types of models for r_t : (i) equilibrium models and (ii) no-arbitrage models.

    Equilibrium models start from assumptions about economic variables and produce a process for r_t. This modelling approach then explores the implications to prices of bond, options and other term-structure derivatives. Examples are the Cox-Ingersoll-Ross (CIR, 1985) and the Vasiček (1977) models.

    On the other hand, no -arbitrage models are designed to be consistent with today's term structure of interest rates. Examples include the Ho-Lee (1986), Hull-White (1990) and Black-Karasinski (1990) models.

12. The Vasiček (1977) model for r_t is given by the SDE $dr_t = a(b - r_t)dt + \sigma dW_t$, where a, b and σ are positive constants. Here, b is the mean-reverting level, a is the speed of mean reversion, and σ is the volatility. Vasiček model is a version of the so-called Ornstein-Uhlenbeck (OU) process.

    The simplicity of this model is its attractive feature along with its tractable (analytic) solution to the bond pricing problem.

    The drawback of this model though is that r_t is normally distributed. Hence, r_t has a positive probability of hitting 0 or negative values. One may choose appropriate model parameters to minimise of this event happening.

13. An important feature of interest-rate dynamics is mean-reversion. This is a result (to some extent) of the intervention of the regulatory authority such as the Feds or Central Bank. When rates are high, economy tends to slow down and there is low demand for funds from the borrowers. So rates decline.

    On the other hand, when rates are low, there tends to be high demand for funds on the part of the borrowers and rates tend to rise.

14. The CIR (1986) model is given by the SDE $dr_t = a(b-r_t)dt+\sigma\sqrt{r_t}dW_t$. This model was proposed to rectify the deficiency (negative rates) of the Vasiček model. The volatility component contains $\sqrt{r_t}$, which does not permit negative values for r_t. This model still has mean-reversion, but the density is a non-central chi square.

    In particular, the CIR model is derived from the sum of squares of OU processes. That is, $r_t^2 := (X_t^1)^2 + \ldots + (X_t^n)^2$, where each X_t^i has dynamics $dX_t^i = -\frac{\alpha}{2}X_t^i dt + \frac{\sigma}{2}dW_t^i$ with α and σ being positive constants and X_0^i is known.

15. The Ho-Lee (1984) model is specified by the SDE $dr_t = \theta(t)dt + \sigma dW_t$. The function $\theta(t)$ is chosen in such a way that the model fits the initial term structure. It may be shown that the function $\theta(t)$ can be calculated from the initial term structure through the forward rate.

16. The Hull-White (1990) model is an extended Vasiček model. It is a Vasiček model with time-varying parameters, i.e.,

$$dr_t = a(t)(b(t) - r_t)dt + \sigma(t)dW_t.$$

It offers more flexibility being able to fit more shapes of the yield curve compared to the capability of the usual Vasiček model.

17. The Vasiček, Ho-Lee, Hull-White and CIR models have analytic bond price solution. Their bond pricing solutions fall under the category of **exponential affine** forms in r_t. This means that under these models $B(t,T) = \exp(-r_t A(t,T) + C(t,T))$, where $A(t,T)$ and $C(t,T)$ are deterministic functions of t, T and the specified model parameters.

18. The Black-Karasinski (1990) model attempts to rectify the weakness of both Vasiček and Ho-Lee models, which is the possibility of getting negative or zero rates. Under this model, $d \ln r_t = (\theta(t) - a(t) \ln r_t)dt + \sigma(t)dW_t$. That is, the value of r_t is log-normal. There is no analytic solution for the bond price under this model setting.

19. The Black-Derman-Toy (1990) model has the same r_t specification as that of the Black-Karasinski model (i.e., log-normal r_t) but with the condition that $a(t) = -\dfrac{\sigma'(t)}{\sigma(t)}$. Here, $a(t)$ is the mean-reversion rate, which is only positive if volatility of the short rate is a decreasing function of time.