SS4521G - 17–21 March 2014

SUMMARY OF IMPORTANT POINTS DISCUSSED IN THE LECTURE

The following concepts/theories were covered/reviewed:

1. After the discussion of the taxonomy of exotic derivatives, we looked at closely the hedging and pricing of Asian options.

2. The analysis of Asian options entails the multi-dimensional Feynman-Kac (FK) theorem. The FK theorem for the 2-dim case (2 SDEs driven by 2 BMs) is as follows.

   Let $W(t) = (W_1(t), W_2(t))$ be a two-dimensional Brownian, that is, a vector of two independent, one-dimensional Brownian motions. Consider two SDEs

$$\begin{aligned} dX_1(t) &= \beta_1(t, X_1(t), X_2(t))dt + \gamma_{11}(t, X_1(t), X_2(t))dW_1(t) \\ &\quad +\gamma_{12}(t, X_1(t), X_2(t))dW_2(t) \\ dX_1(t) &= \beta_2(t, X_1(t), X_2(t))dt + \gamma_{21}(t, X_1(t), X_2(t))dW_1(t) \\ &\quad +\gamma_{22}(t, X_1(t), X_2(t))dW_2(t). \end{aligned}$$

   The solution of this pair of SDEs depends on the initial values $X_1(t) = x_1$ and $X_2(t) = x_2$. Regardless, of the initial condition, the solution is a Markov process.

   Furthermore, consider the function $h(y_1, y_2)$. Corresponding to the initial condition t, x_1, x_2, where $0 \le t \le T$, define the conditional expectations

$$\begin{aligned} g(t, x_1, x_2) : &= E_{t,x_1,x_2} h(X_1(T), X_2(T)), \\ f(t, x_1, x_2) : &= E_{t,x_1,x_2} \left[e^{-r(T-t)} h(X_1(T), X_2(T)) \right]. \end{aligned}$$

Then

$$
\begin{aligned}
&g_t + \beta_1 g_{x_1} + \beta_2 g_{x_2} + \frac{1}{2}(\gamma_{11}^2 + \gamma_{12}^2) g_{x_1 x_1} \\
&(\gamma_{11}\gamma_{21} + \gamma_{12}\gamma_{22}) g_{x_1} g_{x_2} + \frac{1}{2}(\gamma_{21}^2 + \gamma_{22}^2) g_{x_2 x_2} = 0 \qquad (1)\\
&f_t + \beta_1 f_{x_1} + \beta_2 f_{x_2} + \frac{1}{2}(\gamma_{11}^2 + \gamma_{12}^2) f_{x_1 x_1} \\
&(\gamma_{11}\gamma_{21} + \gamma_{12}\gamma_{22}) f_{x_1} f_{x_2} + \frac{1}{2}(\gamma_{21}^2 + \gamma_{22}^2) f_{x_2 x_2} = rf. \qquad (2)
\end{aligned}
$$

The functions g and f satisfy the terminal conditions $g(T, x_1, x_2) = f(T, x_1, x_2) = h(x_1, x_2)$ for all x_1 and x_2.

Remark: The outline of the proof for the above theorem was given in the lecture.

3. Equation (2) is called the Dicounted Feynman-Kac theorem, which can be used to find prices and hedges, even for path-dependent options. We started to demonstrate this by analysing an Asian option.

4. The payoff for an Asian option is

$$V(T) = \left(\frac{1}{T}\int_0^T S(u) - X\right)^+,$$

where $S(u)$ is a geometric Brownian motion, the expiration time T is fixed and X is the strike price. It is assumed that $S(u)$ is a geometric Brownian motion under a risk-neutral measure.

5. The Asian option's payoff depends on the whole path of the stock price via its integral. That is, we must know $Y(t) := \int_0^t S(u)du$. The Y

process has the differential form $dY(u) = S(u)du$.

6. The above **pair** of processes $(S(u), Y(u))$ is a 2-dim Markov processes. Note that $Y(u)$ alone is not a Markov process because its SDE involves the process $S(u)$. However, the pair $(S(u), Y(u))$ is Markov because their pair of SDEs involves only these processes and of course, the driving Brownian motion for $S(u)$.

7. From the risk-neutral pricing and noting the Markov property of the pair $((S(t), Y(t))$, the value of the Asian option at time t, $t < T$ can be written as

$$v(t, S(t), Y(t)) = V(t) = E\left[e^{-r(T-t)} \left(\frac{1}{T} Y(T) - X \right) \middle| \mathcal{F}_t \right]$$

subject to the terminal condition $v(T, x, y) = \left(\frac{y}{T} - X \right)^+$ for all x and y.

To find the hedging strategy, we shall look at the PDE satisfied by $v(t, x, y)$ in the next lecture.