Actuarial Research Group Seminar Notes

Preservation under mixing of the D-CM, the D-LCVX, and the D-CG classes of probability distributions under mixing

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$$\begin{tikzcd}
D-CM \arrow[dr] & & \\
& D-LCVX \arrow[dr] & \\
& & D-CG
\end{tikzcd}$$

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1 Introduction

In this note we consider some discrete reliability classes. In Section 2 we provide their definitions. Further, they are known to satisfy certain inclusion properties, which are visualized in Figures 1 and 2 where an arrow denotes that the left-hand class is a nontrivial subclass of the right-hand one. (These graphs are reproduced from Willmot and Cai, 2001, and Pavlova et al., 2006.) As it is seen in Table 1 (also reproduced from Pavlova et al., 2006), for most of these classes it is known whether they are preserved under either mixing or convolution. In this note we prove that the D-CM and the D-LCVX classes are closed under mixing. It is known that the D-CG class is not preserved under convolution (by a personal communication with G. Willmot) but it remains to be verified whether this class is closed under mixing.

CLASS	PRESERVED UNDER	CLASS	PRESERVED UNDER
D-CM			
D-LCVX		D-LCVE	convolution
D-CG			
DS-DFR	mixing	DS-IFR	convolution
D-DFR	mixing	D-IFR	*not* preserved under convolution
DS-NWU	mixing (no crossing)	D-NBU	convolution
D-DFRA	mixing	D-IFRA	convolution
D-NWU	mixing (no crossing)	DW-NBU	convolution
D-IMRL	mixing	D-DMRL	*not* preserved under convolution
DS-NWUC	mixing (no crossing)	D-NBUC	convolution
D-NWUC	mixing (no crossing)	DW-NBUC	convolution
DS-NWUE	mixing (no crossing)	D-NBUE	convolution
D-NWUE	mixing (no crossing)	DW-NBUE	convolution
D-HNWUE	mixing	D-HNBUE	convolution

Table 1: Preservation of discrete classes under mixing and convolution

2 Definitions of discrete reliability classes

Let $F(x) = 1 - \bar{F}(x)$, $x \in \mathbb{N}$, be a discrete c.d.f. with corresponding probability mass function (p.f.) f.

Definition 2.1. *The distribution F is called* DISCRETE COMPLETELY MONOTONE *or D-CM if* $(-1)^n \Delta^n f(x) \geq 0$ *for* $n, x \in \mathbb{N}$, *where* $\Delta f(x) = f(x+1) - f(x)$, $\Delta^0 f(x) = f(x)$, *and* $\Delta^n = \Delta\left(\Delta^{n-1}\right)$.

Definition 2.2. *The distribution F is called* DISCRETE LOG-CONVEX (DISCRETE LOG-CONCAVE) *or D-LCVX (D-LCVE) if* $f^2(x) \leq (\geq) f(x-1)f(x+1)$ *for* $x \in \mathbb{N}_+$.

Definition 2.3. *The distribution F is called* DISCRETE COMPOUND GEOMETRIC *or D-CG if its probability generating function* $\mathcal{F}$ *satisfies* $\mathcal{F}(z) = [1-\varphi]/[1-\varphi\mathcal{Q}(z)]$ *for some* $\varphi \in (0,1)$ *and some probability generating function* $\mathcal{Q}$.

Definition 2.4. *The distribution F is called* DISCRETE DECREASING FAILURE RATE (DISCRETE INCREASING FAILURE RATE) *or D-DFR (D-IFR) if its failure rate* $h_F(x) = f(x)/[f(x) + \bar{F}(x)]$ *is non-decreasing (non-increasing) for* $x \in \mathbb{N}_+$.

Definition 2.5. *The distribution F is called* DISCRETE STRONGLY DECREASING FAILURE RATE (DISCRETE STRONGLY INCREASING FAILURE RATE) *or DS-DFR (DS-IFR) if its failure rate* $h_F(x) = f(x)/[f(x) + \bar{F}(x)]$ *is non-decreasing (non-increasing) for* $x \in \mathbb{N}$.

Definition 2.6. *The distribution F is called* DISCRETE NEW WORSE THAN USED (DISCRETE NEW BETTER THAN USED) *or D-NWU (D-NBU) if* $\bar{F}(x+y) \geq (\leq)\, \bar{F}(x)\bar{F}(y)$, $x, y \in \mathbb{N}$.

Definition 2.7. *The distribution F is called* DISCRETE STRONGLY NEW WORSE THAN USED (DISCRETE WEAKLY NEW BETTER THAN USED) *or DS-NWU (DW-NBU) if* $\bar{F}(x+y+1) \geq (\leq)\, \bar{F}(x)\bar{F}(y)$, $x, y \in \mathbb{N}$.

Definition 2.8. *The distribution F is called* DISCRETE DECREASING FAILURE RATE AVERAGE (DISCRETE INCREASING FAILURE RATE AVERAGE) *or D-DFRA (D-IFRA) if* $[\bar{F}(x)]^{1/x}$ *is non-increasing (non-decreasing) in* $x \in \mathbb{N}_+$.

Definition 2.9. *The distribution F is called* DISCRETE INCREASING MEAN RESIDUAL LIFETIME (DISCRETE DECREASING MEAN RESIDUAL LIFETIME) *or D-IMRL (D-DMRL) if its mean residual lifetime* $r_F(x) = \sum_{i=x}^{\infty} \bar{F}(i)/\bar{F}(x)$ *is non-decreasing (non-increasing) in* $x \in \mathbb{N}$.

Definition 2.10. *The distribution F with expectation* $\sum_{i=0}^{\infty} \bar{F}(i) = \theta > 1$ *is called* DISCRETE HARMONIC NEW WORSE THAN USED IN EXPECTATION (DISCRETE HARMONIC NEW BETTER THAN USED IN EXPECTATION) *or D-HNWUE (D-HNBUE) if* $\sum_{i=x}^{\infty} \bar{F}(i) \geq (\leq)\, \theta(1-1/\theta)^x$, $x \in \mathbb{N}_+$.

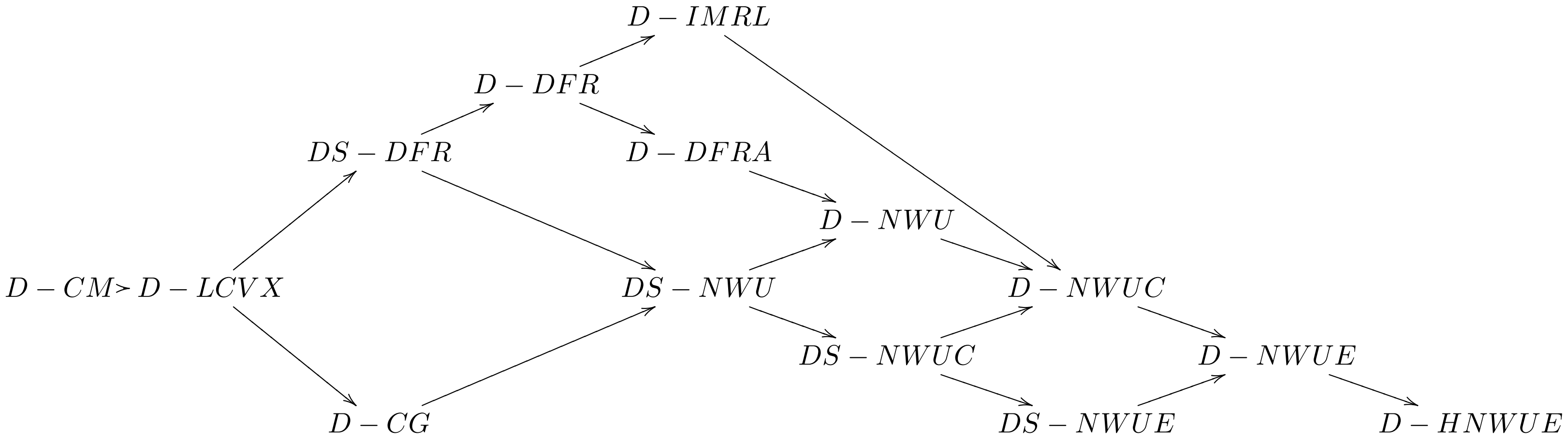

Figure 1: Inclusion relationships of the classes

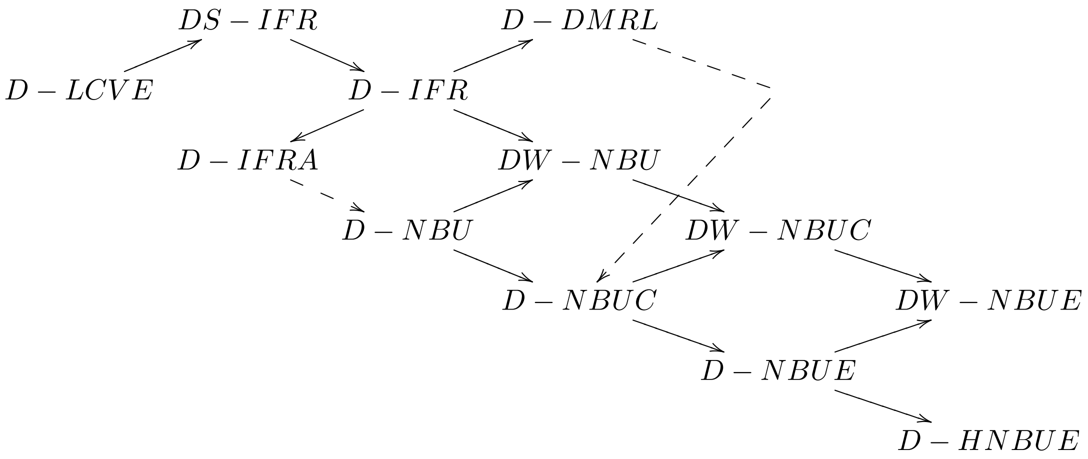

Figure 2: Inclusion relationships of the dual classes

3 Preservation of the D-CM and the D-LCVX classes under mixing

The following proof is based on an idea of Bruce Jones:

Proposition 3.1. *The D-CM class of probability distributions is closed under mixing.*

Proof. Let g be some discrete probability distribution. Without loss of generality we may assume that its support is on the positive integers $\mathbb{N}_+$. We will employ an alternative definition of the D-CM class that is provided by Lemma 1.5.11 of Van Harn (1978). Namely, the functions $f_j,\ j = 1, 2, \ldots,$ belong to the D-CM class, if and only if

$$f_j(x) = \int_0^1 (1-p)p^x dB_j(p), \quad x \in \mathbb{N}, \tag{1}$$

where B_j is a probability distribution on $[0, 1)$. Then, the mixture $f(x) = \sum_{j=1}^{\infty} f_j(x)g(j)$ satisfies

$$f(x) = \int_0^1 (1-p)p^x d\left[\sum_{j=1}^{\infty} g(j)B_j(p)\right], \quad x \in \mathbb{N},$$

where the mixture $\sum_{j=1}^{\infty} g(j)B_j(p)$ is also a probability distribution on $[0, 1)$. Thus, the required result follows. $\square$

The following proof is due to Min Jiang:

Lemma 3.2. *A linear combination of two functions in the D-LCVX class is a function from the same class.*

Proof. Assuming $f_1(x)$ and $f_2(x)$ are two probability functions in the D-LCVX class, it follows by the definition of D-LCVX that,

$$f_1^2(x) \leq f_1(x-1)\, f_1(x+1) \tag{2}$$

$$f_2^2(x) \leq f_2(x-1)\, f_2(x+1) \tag{3}$$

Let $f(x)$ be a mixture of $f_1(x)$ and $f_2(x)$, which is defined as,

$$f(x) = \alpha\, f_1(x) + \beta\, f_2(x) \tag{4}$$

with $\alpha + \beta = 1$. We want to show that

$$f^2(x) \leq f(x-1)\, f(x+1) \tag{5}$$

The left hand side of inequality (5) can be expanded as,

$$\begin{aligned} f^2(x) &= (\alpha\, f_1(x) + \beta\, f_2(x))^2 \\ &= \alpha^2\, f_1^2(x) + \beta^2 f_2^2(x) + 2\,\alpha\,\beta f_1(x) f_2(x)\,. \end{aligned}$$

The right hand side of inequality (5) can be expanded as,

$$\begin{aligned} f(x-1)\, f(x+1) &= (\alpha\, f_1(x-1) + \beta\, f_2(x-1))\,(\alpha\, f_1(x+1) + \beta\, f_2(x+1)) \\ &= \alpha^2 f_1(x-1)\, f_1(x+1) + \beta^2 f_2(x-1)\, f_2(x+1) \\ &\quad +\alpha\beta f_1(x-1)\, f_2(x+1) + \alpha\beta f_1(x+1)\, f_2(x-1)\,. \end{aligned}$$

On making use of the properties specified by inequality (2) and (3), we know that,

$$\alpha^2\, f_1^2(x) + \beta^2 f_2^2(x) \le \alpha^2 f_1(x-1)\, f_1(x+1) + \beta^2 f_2(x-1)\, f_2(x+1)\,.$$

In order to prove inequality (5), we only need to show

$$2\,\alpha\,\beta f_1(x) f_2(x) \le \alpha\beta f_1(x-1)\, f_2(x+1) + \alpha\beta f_1(x+1)\, f_2(x-1),$$

which is

$$f_1(x-1)\, f_2(x+1) - 2\, f_1(x) f_2(x) + f_1(x+1)\, f_2(x-1) \ge 0. \tag{6}$$

On making use of the properties specified by inequalities (2) and (3), we know that

$$-2\, f_1(x) f_2(x) \ge -2\sqrt{f_1(x-1)\, f_1(x+1)}\sqrt{f_2(x-1)\, f_2(x+1)}\,,$$

so that the left hand side of inequality (6) can be expressed as

$$\begin{aligned} LHS &\ge f_1(x-1)\, f_2(x+1) - 2\sqrt{f_1(x-1)\, f_1(x+1)}\sqrt{f_2(x-1)\, f_2(x+1)} + f_1(x+1)\, f_2(x-1) \\ &= \left(\sqrt{f_1(x-1)\, f_2(x+1)} - \sqrt{f_1(x+1)\, f_2(x-1)}\right)^2 \\ &\ge 0\,. \end{aligned}$$

It follows that inequality (5) holds for any $\alpha + \beta = 1$. The proof for the concave case should be similar to this. □

Using the result of Lemma 3.2, it may be proved by induction that

Proposition 3.3. *The D-LCVX class of probability distributions is closed under mixing.*

The above proposition is also a consequence of the Lemma in Kingman (1961).

It is noteworthy that the alternative definition (1) of the D-CM class implies that all members of this class are mixtures of geometric distributions. Since a geometric distribution is also a D-CG distribution and D-CM $\subset$ D-CG, we conclude that a mixture of geometric distributions may not be used as a counterexample of the closure of the D-CG class under mixing.

References

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