Online Appendix for Portmanteau Tests for ARMA Models with Infinite Variance

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APPENDIX C: THE GENERALIZATION OF LINEAR EXPANSION OF RESIDUAL AUTOCORRELATION

C.1 Introduction

The linear expansion of residual autocorrelations in Box and Pierce (1970) is an approach to deriving the asymptotic distribution of residual autocorrelation functions. Their result was established under the assumption that error sequences have finite variance and the parameters are estimated using least squares. Their expansion may not be valid if the parameters of interest are estimated using other estimation methods or linear processes with infinite variance.

Recently, many researchers suggest using other estimation methods to ARMA models with infinite variance. Therefore, a generalization of linear expansion of residual autocorrelations is needed for using residual autocorrelations as diagnostic tools for ARMA models with infinite variance. This appendix demonstrates that the linear expansion in Box and Pierce (1970) also holds for other estimation methods and for AR models with stable Paretian errors.

C.2 The Autoregressive Process

Consider an $\mathrm{AR}(p)$ process as follows:

$$\phi(B)y_t = a_t, \tag{1}$$

where B denotes the backward operator, $\phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p$, and $\{a_t\}$ is a sequence of IID random variables with mean zero and finite variance σ_a^2. For given values $\dot{\Phi} = \left(\dot{\phi}_1, \cdots, \dot{\phi}_p\right)^T$ of parameters, we can define

$$\dot{a}_t = a_t(\dot{\Phi}) = y_t - \dot{\phi}_1 y_{t-1} - \cdots - \dot{\phi}_p y_{t-p} = \dot{\Phi}(B) y_t \tag{2}$$

and the corresponding autocorrelation function at lag k as

$$\dot{r}_k = r_k(\dot{\Phi}) = \frac{\sum \dot{a}_t \dot{a}_{t-k}}{\sum \dot{a}_t^2}. \tag{3}$$

C.3 Linear Expansion of Residual Autocorrelation Function about Error Autocorrelation Functions

Consider approximating the residual autocorrelation $\hat{r}_k$ by a first order Taylor expansion about $\hat{\Phi} = \Phi$. Let $\dot{c}_k$ and $\dot{r}_k$ denote $\sum \dot{a}_t \dot{a}_{t-k}$ and $\dot{c}_k / \dot{c}_0$ respectively, where $k \in$ integer. Consider the estimators of Φ satisfying

$$\hat{\phi}_j = \phi_j + O_p\left(1/\sqrt{n}\right), \ \forall \ j. \tag{4}$$

We have

$$\hat{r}_k = r_k + \sum_{j=1}^{p} \left(\phi_j - \hat{\phi}_j\right) \hat{\delta}_{jk} + O_p\left(1/n\right), \tag{5}$$

where

$$\begin{aligned} \hat{\delta}_{jk} &= -\frac{\partial \dot{r}_k}{\partial \dot{\phi}_j}|_{\dot{\Phi}=\hat{\Phi}} \\ &= -\frac{\partial}{\partial \dot{\phi}_j}\left(\frac{\dot{c}_k}{\dot{c}_0}\right)|_{\dot{\Phi}=\hat{\Phi}} \\ &= \hat{\delta}_{ij}^{(1)} + \hat{\delta}_{ij}^{(2)}, \end{aligned} \tag{6}$$

$$\hat{\delta}_{ij}^{(1)} = -\dot{c}_k \frac{\partial}{\partial \dot{\phi}_j}\left(\frac{1}{\dot{c}_0}\right)|_{\dot{\Phi}=\hat{\Phi}}$$

and

$$\hat{\delta}_{ij}^{(2)} = -\frac{1}{\dot{c}_0} \frac{\partial \dot{c}_k}{\partial \dot{\phi}_j}|_{\dot{\Phi}=\hat{\Phi}}.$$

For LS estimates, it is straightforward that $\hat{\delta}_{ij}^{(1)} = 0$. Box and Pierce (1970) then showed that $\hat{\delta}_{jk} = \psi_{k-j}$ to order $O_p\left(n^{-1/2}\right)$, where ψ_j's are the impulse response coefficients of the MA (∞) representation of eqn. (1). For other estimation methods, however, $\hat{\delta}_{ij}^{(1)}$ may not be zero. To obtain a general result for $\hat{\delta}_{ij}$, therefore, we will calculate $\hat{\delta}_{ij}^{(1)}$ explicitly.

Note that $\hat{\delta}_{ij}^{(1)}$ can be written as follows:

$$\dot{c}_k \cdot \left[\sum \dot{a}_t^2\right]^{-2} \frac{\partial \dot{c}_0}{\partial \dot{\phi}_j}|_{\dot{\Phi}=\hat{\Phi}}. \tag{7}$$

By eqn. (2.15) of Box and Pierce (1970) and letting $k = 0$, eqn. (7) can be expressed as follows:

$$\begin{aligned} & \frac{\sum y_t^2}{\sum \hat{a}_t^2} \cdot \sum_{i=0}^{p} \hat{\phi}_i \left[r_{-i+j}^{(y)} + r_{i-j}^{(y)}\right] \cdot \frac{\hat{c}_k}{\hat{c}_0} \\ = & \frac{\sum_{i=0}^{p} \hat{\phi}_i \left[r_{-i+j}^{(y)} + r_{i-j}^{(y)}\right]}{\sum_{i=0}^{p} \sum_{j=0}^{p} \hat{\phi}_i \hat{\phi}_j r_{i-j}^{(y)}} \cdot \hat{r}_k, \end{aligned} \tag{8}$$

where

$$r_\nu^{(y)} = \frac{\sum y_t y_{t-\nu}}{\sum y_t^2}.$$

Let $\hat{\zeta}_j$ denote

$$\left(\sum_{i=0}^{p} \hat{\phi}_i \left[r_{-i+j}^{(y)} + r_{i-j}^{(y)}\right]\right) / \left(\sum_{i=0}^{p} \sum_{j=0}^{p} \hat{\phi}_i \hat{\phi}_j r_{i-j}^{(y)}\right),$$

and approximate $\hat{\zeta}_j$ by replacing $\hat{\phi}$'s and $r^{(y)}$'s with ϕ's and ρ's, the theoretical parameters and the autocorrelations of the autoregressive process $\{y_t\}$. By the Barteltt's formula,

$$r_k^{(y)} = \rho_k + O_p\left(1/\sqrt{n}\right)$$

as well as eqn. (4) and (8), we have

$$\hat{\zeta}_j = \zeta_j + O_p\left(1/\sqrt{n}\right). \tag{9}$$

Then by making use of the recursive relation which is satisfied by the autocorrelations of an autoregressive process, eqn. (2.19) of Box and Pierce (1970), or

$$\rho_\nu - \phi_1\rho_{\nu-1} - \cdots - \phi_p\rho_{\nu-p} = \phi(B)\rho_\nu = 0, \quad \nu \geq 1, \tag{10}$$

ζ_j can be simplified to yield

$$\zeta_j = \frac{\sum_{i=0}^p \phi_i\rho_{-j+i}}{\sum_{i=0}^p \phi_i\rho_i}. \tag{11}$$

Note that eqn. (11) has the same form of eqn. (2.20) of Box and Pierce (1970). Specifically, it can be seen as δ_{-j}. Moreover, Box and Pierce indicated that $\delta_\nu = 0,\ \nu < 0$ so $\zeta_j = 0$. Plugging this result into eqn. (8), we have $\hat{\delta}_{ij}^{(1)} = 0$. Consequently, eqn. (2.20) of Box and Pierce (1970) for the linear expansion of residual autocorrelations still holds for other estimators with order $\hat{\phi}_i - \phi = O_p(1/\sqrt{n})$.

Remark 6 : Many estimators of $\phi_{(p)}$ for an AR model with Paretian stable errors have order $O_p([n/\log(n)]^{-1/\alpha})$, such as Whittle's, Yule-Walker and LS estimtors. Using the result that $\mathbf{r}_{(p)} = \rho_{(p)} + O_p([n/\log(n)]^{-1/\alpha})$, and following the arguments in this appendix and in Box and Pierce (1970), we may obtain the linear expansion of residual autocorrelation functions for AR models with stable Paretian errors as in

$$\hat{r}_k = r_k + \sum_{j=1}^p (\phi_j - \hat{\phi}_j)\,\psi_{k-j} + O_p\left([n/\log(n)]^{-2/\alpha}\right), \tag{12}$$

where ψ_j is the impulse response coefficient at lag j and $r_k = \sum Z_t Z_{t-k} / \sum Z_t^2$ is the error autocorrelation at lag k.

Remark 7 : Following the arguments in Lin and McLeod (2007, §5.1), we may extend the above linear expansion to a general ARMA model. A detailed proof of the equality in AR and ARMA models is given in Lin (2006).

REFERENCES

Box, G.E.P. and Pierce, D.A. (1970), "Distribution of Residual Autocorrelation in Autoregressive-Integrated Moving Average Time Series Models," *Journal of American Statistical Association* 65, 1509-1526.

Lin, J.-W. (2006), *Essays on Diagnostic Checks in Time Series.* Ph.D. Thesis, University of Western Ontario.

Lin, J.-W. and McLeod (2007, conditionally accepted), "Portmanteau Tests for ARMA Models with Infinite Variance," *Journal of Time Series Analysis.*

McCulloch, J.H. (1986), "Simple Consistent Estimators of Stable Distribution Parameters," *Communication in Statistics–Computation and Simulation*, 15, 1109–1136.