Statistics 3657 : Example of Bivariate PDF and Marginals

Let

$$f_{X,Y}(x,y) = \begin{cases} \frac{15}{4}x^2 & \text{if } -1 < x < 1 \text{ and } 0 < y < 1 - x^2 \\ 0 & \text{otherwise} \end{cases}$$

The support, say D, or region for which $f_{X,Y}$ is positive is shown in Figure 1.

The marginal pdf of X is

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$$

Thus we see from the support region that $f_X(x) = 0$ if $x < -1$ or $x > 1$. Now consider $-1 < x < 1$. For such an x

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy \\ &= \int_{-\infty}^{0} 0\ dy + \int_{0}^{1-x^2} f_{X,Y}(x,y)dy + \int_{1-x^2}^{\infty} 0\ dy \\ &= \int_{0}^{1-x^2} \frac{15}{4}x^2 dy \\ &= \frac{15}{4}x^2(1-x^2) \end{aligned}$$

Thus

$$f_X(x) = \begin{cases} \frac{15}{4}x^2(1-x^2) & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Next we fnd the marginal of Y. The marginal pdf of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$$

Thus we see from the support region that $f_Y(y) = 0$ if $y < 0$ or $y > 1$. Now consider $0 < y < 1$. For such a y

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx \\ &= \int_{-\sqrt{1-y}}^{\sqrt{1-y}} f_{X,Y}(x,y)dx \end{aligned}$$

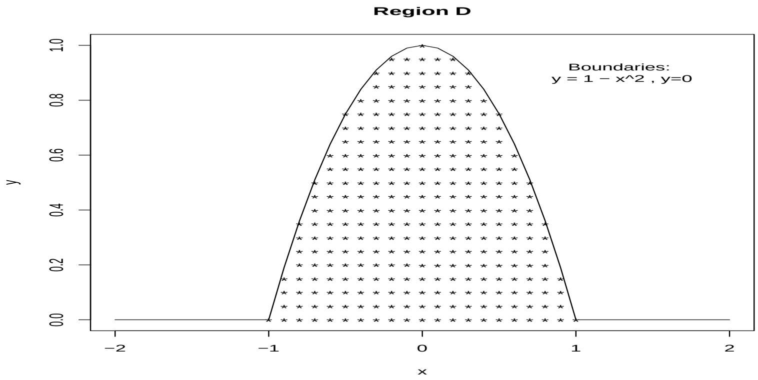

Figure 1: Support of $f_{X,Y}$

$$
\begin{aligned}
&= \int_{-\sqrt{1-y}}^{\sqrt{1-y}} \frac{15}{4} x^2 dx \\
&= \frac{5}{2}(1-y)^{3/2}
\end{aligned}
$$

Thus

$$
f_Y(y) = \begin{cases} \frac{5}{2}(1-y)^{3/2} & \text{if } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}
$$

Remark : The limits of integration are determined by the support of $f_{X,Y}$. The actual algebraic form of this function does not play any role in determining the limits of integration. The limits of integration may often be found conveniently by algebra or geometry. Most students will find geometry and pictures very helpful in finding the limits of integration. This may require practice in sketching regions and the image under some transformation. Thus one of the *Prerequisites Review* problems is about such a topic.

Are X and Y independent? If they are then

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)$$

for all x and y, except for points on the boundary, which do not affect integration. This property does not hold. For example, for $x = -.9$, and $y = .3 > (1 - .9^2) = .19$, we have $f_{X,Y}(x,y) = 0$ (see the support of $f_{X,Y}$), but $f_X(x)f_Y(y) \neq 0$. Therefore X and Y are dependent.

Another useful property of the a pdf is that for sets B we have

$$P((X,Y) \in B) = \int\int_B f(x,y)dydx$$

Example

Consider calculating $P(X \leq Y)$ in our example above. First consider the set $B = \{(x,y) : x \leq y\}$ so that

$$P(X \leq Y) = P((X,Y) \in B)$$

Figure 2 gives the region B intersected with the support of $f_{X,Y}$. It is divided into two parts. Region B_1 includes the part for which $x < 0$. Region B_2 has "corners" at $(0,0)$, $(0,1)$ and (x_0, x_0) where x_0 is the point

$$(x_0, x_0) = (x_0, 1 - x_0^2)$$

on the intersection of the line $y = x$ and the curve $y = 1 - x^2$. Thus x_0 is a root of $x = 1 - x^2$, or equivalently a root of

$$x^2 + x - 1 = 0 \ .$$

Also for this point $x_0 > 0$, so we are searching for a positive root. Solving this quadratic we find

$$x_0 = \frac{-1 + \sqrt{5}}{2} \ .$$

By symmetry

$$\int\int_{B_1} f(x,y)dydx = \frac{1}{2}$$

By direct calculation

$$\begin{aligned}
\int\int_{B_2} f(x,y)dydx &= \int_0^{x_0}\int_x^{1-x^2} \frac{15}{4}x^2 dydx \\
&= \int_0^{x_0} \frac{15}{4}x^2(1 - x^2 - x)dydx \\
&= \frac{-53 + 25\sqrt{5}}{32} \\
&= 0.0906781
\end{aligned}$$

Thus $P(X < Y) = 0.591$ (rounded to 3 digits).

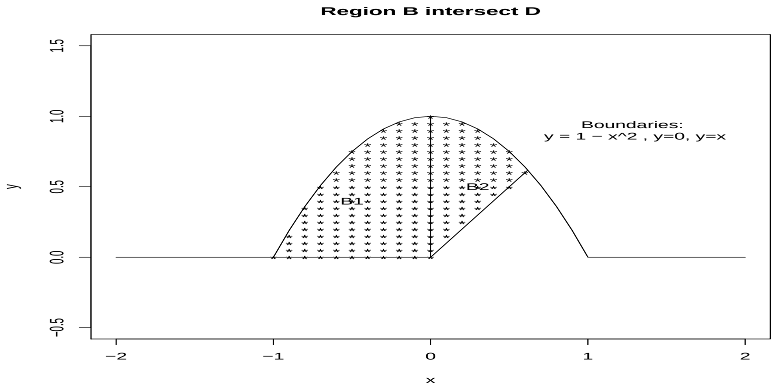

Figure 2: Graph of Region $B = B_1 \cup B_2$

Conditional pdf

The conditional pdf of Y given $X = x$ is

$$f_{Y|X=x}(y) = \begin{cases} \frac{f_{X,Y}(x,y)}{f_X(x)} & \text{if } x \in \text{Support of } f_X \\ \text{undefined} & \text{otherwise} \end{cases}$$

Thus for $-1 < x < 1$ we obtain

$$\begin{aligned} f_{Y|X=x}(y) &= \begin{cases} \frac{\frac{15}{4}x^2}{\frac{15}{4}x^2(1-x^2)} & \text{if } 0 < y < 1 - x^2 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{1}{1-x^2} & \text{if } 0 < y < 1 - x^2 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Notice this says that, for $-1 < x < 1$, $Y|X = x \sim \text{Unif}(0, 1 - x^2)$.

Next we obtain the conditional pdf of X given $Y = y$. Note that this is only defined for $0 < y < 1$. For $0 < y < 1$

$$\begin{aligned} f_{X|Y=y}(x) &= \begin{cases} \frac{\frac{15}{4}x^2}{\frac{5}{2}(1-y)^{3/2}} & \text{if } 0 < y < 1 - x^2 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{3}{2}\frac{x^2}{(1-y)^{3/2}} & \text{if } 0 < y < 1 - x^2 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Moments of X and Y

$$\begin{aligned}
\mathrm{E}(X) &= \int_{-1}^{1} x\frac{15}{4}x^2(1-x^2)dx = 0 \\
\mathrm{E}(X^2) &= \int_{-1}^{1} x^2\frac{15}{4}x^2(1-x^2)dx = \frac{3}{7} \\
\mathrm{E}(Y) &= \int_{0}^{1} y\frac{5}{2}(1-y)^{3/2}dy = \frac{2}{7} \\
\mathrm{E}(Y^2) &= \int_{0}^{1} y^2\frac{5}{2}(1-y)^{3/2}dy = \frac{8}{63}
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}(XY) &= \int_{-1}^{1}\int_{0}^{1-x^2} xy\frac{15}{4}x^2 dydx \\
&= \frac{15}{8}\int_{-1}^{1} x(1-x^2)^2x^2dx \\
&= 0
\end{aligned}$$

Therefore

$$\mathrm{Cov}(X,Y) = \mathrm{E}(XY) - \mathrm{E}(X)\mathrm{E}(Y) = 0$$

Note here that X and Y have 0 covariance and hence are uncorrelated (ie have 0 correlation), but they are dependent.

This furnishes an example of a pair of dependent random variables that are uncorrelated.

In this example we next obtain some conditional expectations in this bivariate example. Recall

$$f_{X|Y=y}(x) = \begin{cases} \frac{3}{2}\frac{x^2}{(1-y)^{3/2}} & \text{if } 0 < y < 1 - x^2 \\ 0 & \text{otherwise} \end{cases}$$

Thus for $0 < y < 1$ we have

$$\begin{aligned} \mathrm{E}(X|Y=y) &= \int_{-\infty}^{\infty} x f_{X|Y=y}(x)dx \\ &= \int_{x:-1<x<1 \text{ and } y<1-x^2} x\frac{3}{2}\frac{x^2}{(1-y)^{3/2}}dx \\ &= \int_{-\sqrt{1-y}}^{\sqrt{1-y}} \frac{3}{2}\frac{x^3}{(1-y)^{3/2}}dx \\ &= 0 \end{aligned}$$

Also

$$\begin{aligned} \mathrm{E}(X^2|Y=y) &= \int_{-\infty}^{\infty} x^2 f_{X|Y=y}(x)dx \\ &= \int_{-\sqrt{1-y}}^{\sqrt{1-y}} \frac{3}{2}\frac{x^4}{(1-y)^{3/2}}dx \\ &= \frac{2}{5}(1-y)^{5/2} \end{aligned}$$

Thus we obtain the two random variables

$$\begin{aligned} \mathrm{E}(X|Y) &= 0 \\ \mathrm{E}(X^2|Y) &= \frac{2}{5}(1-Y)^{5/2} \end{aligned}$$

We could obtain other random variables $\mathrm{E}(g(X)|Y)$ for various other functions g.

At home the student should verify that $\mathrm{E}(\mathrm{E}(X|Y)) = \mathrm{E}(X)$ and $\mathrm{E}(\mathrm{E}(X^2|Y)) = \mathrm{E}(X^2)$. Here we have two random variables that are functions of Y but have expectation equal to the expectations of certain functions of X.