Integrals Over $(-\infty, \infty)$

$$A = \int_{-\infty}^{\infty} f(x)dx = \lim_{a\to-\infty, b\to\infty} \int_a^b f(x)dx$$

For A to be well defined we need the same limit over any sequence $a_n \to -\infty$ and $b_n \to \infty$. In particular for the integral to be well defined it is sufficient for

$$\int_{-\infty}^{\infty} |f(x)|dx < \infty$$

Consider the following example $f(x) = xg(x)$ where

$$g(x) = \begin{cases} \frac{1}{4} & \text{if } |x| \leq 1 \\ \frac{1}{4x^2} & \text{if } |x| > 1 \end{cases}$$

This is the pdf of the ratio of independent Uniform(-1,1) r.v.'s as discussed in an earlier example.

For $a < -1, b > 1$

$$\int_a^b f(x)dx = \int_{-1}^{1} \frac{x}{4}dx - \int_1^{|a|} \frac{1}{4x}dx + \int_1^b \frac{1}{4x}dx = \frac{1}{4}\log(b/|a|)$$

Let $\alpha \in R$ be an arbitrary number. Choose sequences $a_n \to -\infty$, $b_n \to \infty$ such that $a_n < -1$, $b_n > 1$ and

$$\frac{1}{4}\log(b_n/|a_n|) = \alpha \ .$$

This says that

$$a_n = -b_n e^{-4\alpha} \ .$$

Consider now n sufficiently large so that $a_n < -1$.

Then

$$\begin{aligned} A_n &= \int_{a_n}^{b_n} f(x)dx \\ &= \frac{1}{4}\log(b/|a|) \\ &= \alpha \end{aligned}$$

For such a sequence we obtain $\lim A_n = \alpha$ which can be any real number we wish. That is there is no well defined limit A for this integral. In such a case we say that $\int_{-\infty}^{\infty} f(x)dx$ does not exist.

Another example for which the intergral is undefined is

$$\int_{-\infty}^{\infty} x\frac{1}{\pi(1+x^2)}dx$$

so that we say a Cauchy distribution does not have a finite expectation (or first moment).

Finiteness of an Integral

How can we tell if an integral is finite. This is a topic that is studied in first year calculus, usually using an integral test of some form. The student should review this material, perhaps by going to the library and reviewing an introductory calculus text.

Here we will consider only a specific example. Otherwise in this course we will take the finiteness of certain integrals for granted. This example is given to remind the student of these methods, which then can be referred to in class and left for a student to determine if certain integrals discusses in class are finite.

Related to the normal distribution we will be interested in showing that

$$\int_0^\infty x^k e^{-\frac{x^2}{2}} dx < \infty \text{ and } \int_{-\infty}^0 |x|^k e^{-\frac{x^2}{2}} dx < \infty \ .$$

For simplicity we now only consider the first of these two, and only for the case $k = 1$.

Consider the function $g(x) = xe^{-x^2/2}$. Notice that

$$g'(x) = (1 - x^2)e^{-\frac{x^2}{2}}$$

so that g increases from 0 to 1 and then decreases. Next consider the ratio

$$\frac{g(x)}{e^{-x}} = xe^{-\frac{1}{2}x^2 + x}$$

The exponent part is

$$\begin{aligned} -\frac{1}{2}x^2 + x &= -x(\frac{x}{2} - 1) \\ &\leq -x(\frac{4}{2} - 1) \ \text{ if } x \geq 4 \\ &= -2x \end{aligned}$$

Thus for $x \geq 4$ we have

$$g(x) \leq xe^{-2x} \ .$$

Thus

$$\begin{aligned} \int_0^\infty xe^{-\frac{1}{2}x^2} dx &= \int_0^4 xe^{-\frac{1}{2}x^2} dx + \int_4^\infty xe^{-\frac{1}{2}x^2} dx \\ &\leq \int_0^4 x dx + \int_4^\infty xe^{-2x} dx \end{aligned}$$

using the bound above for the second integral and using the fact that $\exp\{-x^2/2\} \leq 1$ on $[0, 4]$. We then obtain

$$\begin{aligned} \int_0^\infty xe^{-\frac{1}{2}x^2} dx &\leq \frac{4^2}{2} + \int_0^\infty xe^{-2x} dx \\ &= 8 + \frac{1}{4}\int_0^\infty ue^{-u} du \\ &= 8 + \frac{1}{4}\Gamma(2) \\ &= 8 + \frac{1}{4} \end{aligned}$$

Thus

$$\int_0^\infty xe^{-\frac{1}{2}x^2} dx < \infty \ .$$