Ratio of Two Independent Unif(-1,1) Random Variables

1 General Comments

Suppose that X, Y are iid Uniform(-1,1) random variables. Let $Z = \frac{Y}{X}$.

In this handout we find the pdf of Z using two methods. One is the method of the special form derived for the ratio. The other method is completing the transformation to construct a one to one invertible differentiable transform and then find the marginal as needed.

2 Using the Special Form of Quotient PDF

Using the special derived form we know the integral that needs to be calculated. In this section we carefully do this calculation.

Thus the pdf of Z is

$$f_Z(z) = \int_{-\infty}^{\infty} |x| f_X(x) f_Y(xz) dx \ . \tag{1}$$

We need to pay attention to the region of integration in (1). In particular we have to integrate over the set

$$B = \{x \ : \ -1 < x < 1 \text{ and } -1 < xz < 1\}$$

This set comes from the supports of the two pdf's f_X and f_Y. Notice also that the set B depends on the particular value of z, the argument of the function f_Z. We may therefore wish to use a notation B_z to remind us that the set B depends on this value z. This might be helpful for students when working through some of these problems, but with practice you will automatically note this. Thus we use the notation B for this set.

Care has to be taken to manipulate the inequalities in the set B. In particular this suggests we should consider two cases, $z < 0$ and $z > 0$.

Consider first $z < 0$.

$$\begin{aligned}
& -1 < xz < 1 \\
\Leftrightarrow \quad & -1 < xz \text{ and } xz < 1 \\
\Leftrightarrow \quad & x < -\frac{1}{z} = \frac{1}{|z|} \text{ and } x > \frac{1}{z} = -\frac{1}{|z|} \\
& \text{notice that } \frac{1}{z} \text{ is negative} \\
\Leftrightarrow \quad & -\frac{1}{|z|} < x < \frac{1}{|z|}
\end{aligned}$$

Thus the set B taking into account the intersection of the two pieces is given by

$$B = \begin{cases} \{x \ : \ -1 < x < 1\} & \text{if } |z| < 1, \\ \{x \ : \ -\frac{1}{|z|} < x < \frac{1}{|z|}\} & \text{if } |z| \geq 1. \end{cases}$$

Next consider $z > 0$. Then

$$\begin{aligned}
&-1 < xz < 1 \\
&\quad\Leftrightarrow\quad -1 < xz \text{ and } xz < 1 \\
&\quad\Leftrightarrow\quad x > -\frac{1}{z} \text{ and } x < \frac{1}{z} \\
&\quad\Leftrightarrow\quad -\frac{1}{z} < x < \frac{1}{z} \\
&\quad\Leftrightarrow\quad -\frac{1}{|z|} < x < \frac{1}{|z|}
\end{aligned}$$

This last line is only for convenience as we can write it in the same form as the $z < 0$ case. It is not needed except for convenience.

Thus the set B taking into account the intersection of the two pieces is given by

$$B = \begin{cases} \{x \;:\; -1 < x < 1\} & \text{if} |z| < 1, \\ \{x \;:\; -\frac{1}{|z|} < x < \frac{1}{|z|}\} & \text{if} |z| \geq 1. \end{cases}$$

Thus

$$\begin{aligned}
f_Z(z) &= \int_{-\infty}^{\infty} |x| f_X(x) f_Y(xz) dx \\
&= \begin{cases} \int_{-1}^{1} |x| f_X(x) f_Y(xz) dx & \text{if } |z| < 1 \\ \int_{-\frac{1}{|z|}}^{\frac{1}{|z|}} |x| f_X(x) f_Y(xz) dx & \text{if } |z| \geq 1 \end{cases} \\
&= \begin{cases} \frac{1}{4} & \text{if } |z| < 1 \\ \frac{1}{4|z|^2} & \text{if } |z| \geq 1 \end{cases}
\end{aligned}$$

The student should now find the pdf in the case X, Y are iid Unif(0,1) (Continuous Uniform distribution on (0,1)).

3 Using a General Transformation Method

We could do the same problem using a more general transformation method. Here again we have X, Y i.i.d. f, where f is the uniform (-1,1) pdf.

The setting we are dealing with $h : R \times R \mapsto R$ given by

$$h(x, y) = \frac{y}{x}$$

with the understanding this is undefined when $x = 0$. This in not a one to one transformation (eg (x, x) maps to 1 for any $x \neq 0$).

Thus we need to do some work to be able to apply a general result for invertible differentiable transforms. This can be done by completing the transform in some way so that the new or extended transform is 1 to 1 and differentiable. There are many ways to do this. For example

$$\begin{aligned} h_1(x, y) &= \frac{y}{x} \\ h_2(x, y) &= y^3 \end{aligned}$$

or

$$\begin{aligned} h_1(x, y) &= \tfrac{y}{x} \\ h_2(x, y) &= y \end{aligned} \qquad (2)$$

Either is of the form

$$(h_1, h_2) : R \times R \mapsto R \times R$$

Note that in either case the original transform h is embedded into this invertible transform; in these examples it corresponds to h_1.

A third (of many) possible completion of the transform is

$$\begin{aligned} h_1(x, y) &= \tfrac{y}{x} \\ h_2(x, y) &= x + y \ . \end{aligned} \qquad (3)$$

We can find the inverse transform by solving for a given (u, v)

$$\begin{aligned} u &= \frac{y}{x} \\ v &= x + y \ . \end{aligned}$$

From the first line we obtain $y = ux$. Substituting into the second we obtain $v = x + xu = x(1 + u)$, and hence

$$y = u\frac{v}{1 + u} \ .$$

These yield

$$\begin{aligned} x &= \frac{v}{1 + u} \\ y &= \frac{uv}{1 + u} \ . \end{aligned}$$

When finding the marginal of U based on the transformation (3) the limits of integration are little more difficult to obtain than finding the marginal of U based on the transformation (2).

Next we note that

$$\begin{aligned} h_1(x, y) &= \frac{y}{x} \\ h_2(x, y) &= y^2 \end{aligned}$$

is not invertible (it is not 1 to 1) so it will not work for our purposes.

For the remainder of this example we will not use the first, but instead use (2). This form is algebraically simpler, but any of the invertible differentiable transforms above will work.

If we define $(u, v) = H(x, y) = (h_1(x, y), h_2(x, y)) = (y/x, y)$ we can solve for (x, y) in terms of (u, v) as $(x, y) = H^{-1}(u, v) = (H_1^{-1}(u, v), H_2^{-1}(u, v)) = (v/u, v)$. The transformation H is then from R^2 to R^2. Notice we find the inverse Jacobian matrix

$$\begin{aligned} J &= \frac{\partial H^{-1}(u,v)}{\partial(u,v)} = \frac{\partial(x,y)}{\partial(u,v)} \\ &= \begin{pmatrix} \frac{\partial(v/u)}{\partial u} & \frac{\partial(v/u)}{\partial v} \\ 0 & \frac{\partial v}{\partial v} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{v}{u^2} & \frac{1}{u} \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Thus we have $|\det(J)| = \frac{|v|}{u^2}$. Notice that formally we also have $J = J_1^{-1}(x, y)|_{(x,y)\to(u,v)}$ where $J_1(x, y) = \frac{\partial(u,v)}{\partial(x,y)}$ with appropriate care and interpretation of (u, v) and (x, y). The notation $J_1^{-1}(x, y)|_{(x,y)\to(u,v)}$ means to calculate $J_1(x, y)$, takes it inverse and the substitute (x, y) as functions of (u, v), or equivalently calculate $J_1(x, y)$, substitute (x, y) as functions of (u, v) and then take the inverse of J_1. Finally notice that one can save a step in the calculation since

$$\det(J(u,v)) = \frac{1}{\det(J(x,y)|_{(x,y)\to(u,v)})}$$

that is calculate the determinant of J_1 and then substitute (x, y) as functions of (u, v).

In keeping with the notation of the text we thus consider the transformation

$$\begin{aligned} U &= h_1(X,Y) &= \tfrac{Y}{X} \\ V &= h_2(Y) &= Y \end{aligned}$$

The pdf of (U, V) is then

$$\begin{aligned} f_{U,V}(u,v) &= f_X(H_1^{-1}(u,v)) f_Y(H_2^{-1}(u,v)) |\det(J)| \\ &= f_X(v/u) f_Y(v) \frac{|v|}{u^2} \\ &= \frac{|v|}{4u^2} \ \mathrm{I}\,(-1 \le v/u \le 1 \text{ and } -1 \le v \le 1) \end{aligned}$$

where $\mathrm{I}(A)$ is the indicator function of the set A.

Next we find the marginal distribution of $U = Y/X$. For a given u we need to integrate over the set

$$A_u = \left\{ v : -1 \le \frac{v}{u} \le 1, \text{ and } -1 \le v \le 1 \right\}$$

With some algebraic care (consider $u < 0$ and $u > 0$ separately) we obtain

$$\begin{aligned} A_u &= \{v : -|u| \le v \le |u|, \text{ and } -1 \le v \le 1\} \\ &= \begin{cases} \{v : -|u| \le v \le |u|\} & \text{if } |u| \le 1 \\ \{v : -1 \le v \le 1\} & \text{if } |u| > 1 \end{cases} \end{aligned}$$

Then

1. for $|u| \leq 1$

$$\begin{aligned} f_U(u) &= \int_{-|u|}^{|u|} \frac{|v|}{4u^2} dv \\ &= \frac{1}{4u^2} 2 \int_0^{|u|} v dv \\ &= \frac{1}{4} \end{aligned}$$

2. and for $|u| > 1$

$$\begin{aligned} f_U(u) &= \int_{-1}^{1} \frac{|v|}{4u^2} dv \\ &= \frac{1}{4u^2} 2 \int_0^{1} v dv \\ &= \frac{1}{4u^2} \end{aligned}$$

Combining we obtain

$$f_U(u) = \begin{cases} \frac{1}{4} & \text{if } |u| \leq 1 \\ \frac{1}{4u^2} & \text{if } |u| > 1 \end{cases}$$

Note this is the same answer (except for notation) that we obtained earlier by the special method for quotients.

It is also an interesting exercise to find the marginal distribution of V. Of course we know the answer, V has a Uniform(-1,1) distribution.

$$f_V(v) = \int_{-\infty}^{\infty} f_{U,V}(u,v)du$$

Thus $f_V(v) = 0$ for $v \leq -1$ and $v \geq 1$.

Now consider $-1 < v < 1$.

For $0 \leq v < 1$ and $u > 0$

$$-1 < \frac{v}{u} < 1 \Leftrightarrow u > v$$

For $0 \leq v < 1$ and $u < 0$

$$-1 < \frac{v}{u} < 1 \Leftrightarrow u < -v$$

For $-1 < v < 0$ and $u > 0$

$$-1 < \frac{v}{u} < 1 \Leftrightarrow u > |v|$$

For $-1 < v < 0$ and $u < 0$

$$-1 < \frac{v}{u} < 1 \Leftrightarrow u < -|v|$$

Thus for $0 \leq v < 1$

$$\begin{aligned} f_V(v) &= \int_v^{\infty} \frac{|v|}{2u^2}du + \int_{-\infty}^{-v} \frac{|v|}{2u^2}du \\ &= \int_v^{\infty} \frac{|v|}{u^2}du \\ &= \frac{1}{2} \end{aligned}$$

and for $-1 < v < 0$

$$\begin{aligned} f_V(v) &= \int_{|v|}^{\infty} \frac{|v|}{2u^2}du + \int_{-\infty}^{-|v|} \frac{|v|}{2u^2}du \\ &= \int_{|v|}^{\infty} \frac{|v|}{u^2}du \\ &= \frac{1}{2} \end{aligned}$$

Putting these together we obtain

$$f_V(v) = \begin{cases} \frac{1}{2} & \text{if } -1 < v < 1 \\ 0 & \text{otherwise} \end{cases}$$

Thus we do in fact obtain the answer that V has a Uniform(-1,1) distribution.

4 Expectation

From the above $U = Y/X$ has pdf

$$f_U(u) = \begin{cases} \frac{1}{4} & \text{if } |u| < 1 \\ \frac{1}{4|u|^2} & \text{if } |u| \geq 1 \end{cases}.$$

Thus $\int_{-\infty}^{\infty} u f_U(u)du$ is undefined. Thus $\mathrm{E}(U)$ is undefined.