Change of Variables and Marginals: Example of Student's t Ratio

Suppose that X and Y are independent r.v.'s, $X \in R$ and $Y > 0$. A specific example below is $X \sim N(0,1)$ and $Y \sim \chi^2_{(n)}$.

Consider the transformation of the r.v.'s

$$\begin{aligned} V &= \frac{X}{\sqrt{Y/a}} \\ W &= Y \end{aligned}$$

where $a > 0$ is a positive number.

We must consider the mapping (or transformation) from$R \times R^+ \mapsto R \times R^+$

$$\begin{aligned} v &= \frac{x}{\sqrt{y/a}} \\ w &= y\ . \end{aligned}$$

This is an invertable mapping with inverse

$$\begin{aligned} x &= v\sqrt{w/a} \\ y &= w \end{aligned}$$

The Jacobian of this transformation is

$$J(v,w) = \frac{\partial(x,y)}{\partial(v,w)} = \begin{pmatrix} \sqrt{w/a} & v\frac{1}{2\sqrt{wa}} \\ 0 & 1 \end{pmatrix}$$

and thus

$$\det(J) = \sqrt{w/a}\ .$$

The joint pdf of V, W is then given by

$$f_{V,W}(v,w) = f_{X,Y}(x[v,w], y[v,w])|\det(J)|$$

where $(x[v,w], y[v,w])$ is the inverse map or the formulae of writing x, y in terms of v, w.
Aside: Notice that we should obtain a formula for the pdf $f_{V,W}$ as a function of the arguments v, w. This will be helpful for the student to keep in mind as an aid to obtaining a sensible answer.

Thus we obtain the pdf

$$\begin{aligned} f_{V,W}(v,w) &= f_{X,Y}(x[v,w], y[v,w])\sqrt{w/a} \\ &= f_{X,Y}(v\sqrt{w/a}, w)\sqrt{w/a} \end{aligned}$$

Example

Suppose that $X \sim N(0,1)$, $Y \sim \chi^2_{(n)}$ and that X and Y are independent. Consider the transformation above with $a = n$. Therefore for $v \in R$ and $w > 0$

$$\begin{aligned} f_{V,W}(v,w) &= f_X(v\sqrt{w/n})f_Y(w)\sqrt{w/n}\ , \quad \text{by independence} \\ &= \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\frac{v^2 w}{n}}\frac{1}{2^{n/2}\Gamma\left(\frac{n}{2}\right)}w^{\frac{n}{2}-1}e^{-w/2}\sqrt{w/n} \end{aligned}$$

$f_{V,W}(v,w) = 0$ for all other v, w, that is for $w \leq 0$.

From this we could obtain the marginal pdf of $V = X/\sqrt{Y/n}$. By integration we obtain

$$\begin{aligned}
f_V(v) &= \int_{-\infty}^{\infty} f_{V,W}(v,w)dw \\
&= \int_0^{\infty} f_{V,W}(v,w)dw \\
&= \int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{v^2 w}{n}} \frac{1}{2^{n/2}\Gamma\left(\frac{n}{2}\right)} w^{\frac{n}{2}-1} e^{-w/2} \sqrt{\frac{w}{n}} dw \\
&= \frac{1}{\sqrt{2n\pi}} \frac{1}{2^{n/2}\Gamma\left(\frac{n}{2}\right)} \int_0^{\infty} w^{\frac{n+1}{2}-1} e^{-w\frac{1}{2}\left(\frac{v^2}{n}+1\right)} dw \\
&= \frac{1}{\sqrt{2n\pi}} \frac{1}{2^{n/2}\Gamma\left(\frac{n}{2}\right)} 2^{\frac{n+1}{2}} \left(\frac{v^2}{n}+1\right)^{-\frac{n+1}{2}} \int_0^{\infty} u^{\frac{n+1}{2}-1} e^{-u} du \\
&= \frac{1}{\sqrt{n\pi}} \frac{1}{\Gamma\left(\frac{n}{2}\right)} \left(\frac{v^2}{n}+1\right)^{-\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right) \\
&= \frac{1}{\sqrt{n\pi}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \left(\frac{v^2}{n}+1\right)^{-\frac{n+1}{2}}
\end{aligned}$$