Statistics 4657/9657: Assignment 4

Handout : November 12, 2008.
Due date: November 21, 2008

1. Let X_n have distribution function

$$F_n(x) = x - \frac{\sin(2n\pi x)}{2n\pi} \ ,0 < x < 1.$$

   (a) Show that F_n is indeed a distribution function, and that X_n has a density function. (b) Show that, as $n \rightarrow$, F_n converges to the uniform distribution function, but that the density function of F_n does not converge to the uniform density function.

2. Let $X_1, X_2, \ldots$ be independent variables each taking values +1 or -1 with probabilities $\frac{1}{2}$ each. Show that

$$\sqrt{\frac{3}{n^3}} \sum_{k=1}^{n} kX_k \Rightarrow N(0,1)$$

   as $n \rightarrow \infty$.

   Hint: Use characteristic functions and the Continuity Theorem. Notice this is a CLT for sums of independent r.v.s $Y_k = kX_k$, which are not identically distributed.

3. 7.2.3

4. 7.11.4

5. **Undergraduate students only**
   (a) Compute the characteristic function of a Poisson, λ, distribution. Use this c.f. to find its mean and variance.
   (b) Using c.f. to find the distribution of $X_1 + \ldots + X_n$, where $X_j \sim$ Poisson(λ_j), $j = 1, \ldots, n$ and the X_js are independent. What is your conclusion? Justify it.

6. **For graduate students only** This problem is related to Problem 5.12.40, but does so directly.

   This problem is a special case of the Lindeberg-Feller Central Limit Theorem. The notation used here is the same as in problem 5.12.40.

   (a) Consider
   $$g(x) = e^{ix} = \cos(x) + i\sin(x),\ \ x \in R\ .$$
   The following are facts :
   $$|g(x) - (1 + ix)| \leq \min\{\frac{1}{2}x^2, 2|x|\}$$
   and
   $$|g(x) - (1 + ix - \frac{1}{2}x^2)| \leq \min\{\frac{1}{6}|x|^3, x^2\}\ .$$
   See some comments at the end concerning these inequalities.
   Use these to obtain
   $$|\phi_j(t) - 1| \leq \frac{1}{2}\sigma_j^2 t^2\ .$$
   and
   $$|\phi_j(t) - (1 - \frac{1}{2}t^2\sigma_j^2)| \leq \frac{1}{3}t^3\rho_j\ .$$

   (b) Use Lyapanov's inequality (see page 143) to show that $\sigma_j^3 \leq \rho_j$. Use this to prove that under the condition of the problem that
   $$\frac{1}{\sigma(n)}\max_{1\leq j\leq n}\sigma_j \to 0 \text{ as } n \to \infty\ .$$
   Use this to prove that
   $$\max_{1\leq j\leq n}\left|\phi_j\left(\frac{t}{\sigma(n)}\right) - 1\right| \to 0$$
   as $n \to \infty$.

   (c) Consider the inequality : For complex numbers x_j, y_j with $|x_j| \leq 1$ and $|y_j| \leq 1$
   $$\left|\prod_{j=1}^{n} x_j - \prod_{j=1}^{n} y_j\right| \leq \sum_{j=1}^{n}|x_j - y_j|\ . \tag{1}$$
   Prove this by induction starting with $n = 2$.

   (d) Let $\psi_n(t) = \mathrm{E}(e^{itZ_n})$ be the characteristic function of the random variable
   $$Z_n = \frac{X_1 + X_2 + \ldots + X_n}{\sigma(n)}\ .$$

Verify by part (d) that

$$|\psi_n(t) - e^{-\frac{1}{2}t^2}| \quad \leq \quad \sum_{j=1}^{n} \left| \phi_n\left(\frac{t}{\sigma(n))}\right) - e^{-\frac{\sigma_j^2}{2\sigma(n)^2}t^2} \right| .$$

Prove the following for each real t, and as $n \to \infty$.

i.
$$\sum_{j=1}^{n} \left| \phi_n\left(\frac{t}{\sigma(n))}\right) - \left(1 - \frac{t^2\sigma_j^2}{2\sigma(n)^2}\right) \right| \to 0$$

ii.
$$\sum_{j=1}^{n} \left| \left(1 - \frac{t^2\sigma_j^2}{2\sigma(n)^2}\right) - e^{-\frac{\sigma_j^2}{2\sigma(n)^2}t^2} \right| \to 0$$

iii.
$$|\psi_n(t) - e^{-\frac{1}{2}t^2}| \to 0$$

as $n \to \infty$.

(e) Comment on the inequality in part (a). This is for the student's information and does not require an answer. Consider

$$|g(x) - (1 + ix)| \leq \min\{\frac{1}{2}x^2, 2|x|\} \ . \qquad (2)$$

The constants $\frac{1}{2}$ and 2 come from error bound for complex functions. The remainder term is different for complex Taylor's series is different than for real Taylor's series. However we can obtain (2) but with different constants using just real Taylor's series. The constants do not matter in the context of this type of problem. Below we find such bounds so that the student can understand somewhat how these bounds are obtained.

Using real Taylor's series of order 0 with remainder we have

$$\begin{aligned} \cos(x) &= \cos(0) - \sin(x^*)x = 1 - \sin(x^*)x \\ \sin(x) &= \sin(0) + \cos(x^{**})x = \cos(x^{**})x \end{aligned}$$

where x^* and x^{**} are numbers between 0 and x. Thus

$$\begin{aligned} &|e^{ix} - (1 + ix)| \\ &= |\sin(x^*)x + i(\cos(x^{**}) - 1)x| \\ &= |\sin(x^*) + i(\cos(x^{**}) - 1)||x| \\ &\leq 3|x| \end{aligned}$$

With a first order Taylor's series with remainder we obtain

$$\begin{aligned} \cos(x) &= \cos(0) - \sin(0)x - \cos(x^*)\frac{1}{2}x^2 = 1 - \cos(x^*)\frac{1}{2}x^2 \\ \sin(x) &= \sin(0) + \cos(0)x - \sin(x^*)\frac{1}{2}x^2 = x - \sin(x^*)\frac{1}{2}x^2 \end{aligned}$$

where x^* and x^{**} are numbers between 0 and x.
Thus

$$\begin{aligned} & |e^{ix} - (1 + ix)| \\ = & \ |\cos(x^*)\frac{1}{2}x^2 - i\sin(x^*)\frac{1}{2}x^2| \\ \leq & \ \frac{2}{2}x^2 = x^2 \end{aligned}$$

Since we now have two upper bounds, thus

$$|e^{ix} - (1 + ix)| \leq \min\left(x^2, 3|x|\right) \ .$$