University of Western Ontario
Statistics 3657 Final Exam

18 December 2013: 2 - 5 PM

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Instructions:

I. Make sure that your name and ID number are the front of your exam question sheet. There is also space put your name and ID on each page in case you loosen the staple for the exam booklet.

II. All answers are to be written in this exam booklet, and only this will be graded. A formulae sheet is attached that may be useful for this exam.

III. This exam is of 3 hours duration. This exam consists of 5 questions, and they have approximately equal marks.

IV. Non programmable calculators may be used, but calculators are not needed for this exam.

1	2	3	4	5	TOTAL (100)

Name : ______________________________

ID : ______________________________

1. a) What are the properties of a pdf?

   Is $f(x) = e^{-x}$ a pdf? If yes then verify this and if no verify this.

   b) State the definition of convergence in distribution.

   In the special case of Y_n and Y integer valued r.v.s state an equivalent method, in terms of their pmfs, to prove convergence in distribution of Y_n to the distribution of Y.

   c) State the definition of convergence in probability of a sequence of random variables X_n to a constant a.

   d) State the Central Limit Theorem (CLT), stating specifically the type of convergence. State an example where the CLT may be used in statistics or probability.

e) Suppose X_i are iid exponential parameter λ r.v.s and that $Y_i = X_i^2$. Verify that the Cental Limit Theorem (CLT) applies to the r.v.s Y_i. Then find the limit distribution of

$$\frac{\sqrt{n}(\bar{Y}_n - a)}{b}$$

where a, b are the mean and standard deviation of Y_i. Also find the values of a, b in terms of the parameter λ.

f) State Chebyshev's inequality.

g) State the Continuity Theorem. Discuss in one of two sentences why this theorem in useful in our course.

2. Suppose that X_i, $i \geq 1$ is a sequence of iid random variables, each with a Bernoulli(p) distribution.

   a) Find the moment generating function (mgf) for the Bernoulli distribution.

   b) Let $Y_n = X_1 + \ldots + X_n$. Find the mgf of Y_n and give the distribution of Y_n. State relevant properties or theorems of expectation and mgfs that you are using in your calculations

   c) Use the MGF to obtain the mean and variance of Y_n.

d) Suppose now that $Y_n \sim \text{Binomial}(n, p_n)$ (that is a different p_n for each n), and that $\lim_{n\to\infty} np_n = \lambda > 0$. Show that, for integers $k \geq 0$,

$$\lim_{n\to\infty} P(Y_n = k) = \frac{\lambda^k e^{-\lambda}}{k!} \ .$$

What is the limit distribution of Y_n? Specifically give the name and parameter of this limit distribution.

e) For the remainder of this question p is the same for each n. Let

$$W_n = \sqrt{n}\,(\bar{Y}_n - p)$$

where $\bar{Y}_n = \frac{Y_n}{n}$.

Find the moment generating function of W_n, say M_n.

Find $\lim_{n\to\infty} \log(M_n(t))$.

f) From the calculation in the previous part determine the limit distribution for W_n, stating where, if anywhere, that you use the continuity theorem. Give the limit distribution.

The space below and the back side of this page gives space for the answer to parts e) and f).

3. Consider the joint pdf f of r.v.s (X, Y) given by

$$f(x,y) = \begin{cases} 60x^2(1-y) & \text{if } 0 < x < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

a) Suppose that $U_1, \ldots, U_5$ are iid Uniform(0,1) r.v.s. Show that the pdf f above is the joint pdf of $(U_{(3)}, U_{(4)})$.

b) Sketch the region of the support of f for which also $x < \frac{y}{2}$. Find $P(X < \frac{Y}{2})$.

c) Find the distribution of $W = XY$. Hint : You may do this by finding the cdf (cumulative distribution function) of W or by completing the transformation.

4. The normal distribution and its related distributions are used in many applications in statistics.

   a) Suppose that X, Y have a bivariate normal distribution (see formula sheet).

      - Obtain the marginal distribution of X.
      - Obtain the conditional pdf of Y given that $X = x$. Using this obtain the r.v. $\mathrm{E}(Y|X)$.
      - What is the distribution of $E(Y|X)$ in this case?

b) Suppose $X_1, \ldots, X_n$ are iid $N(\mu, \sigma^2)$.

In this part of the problem you may use the following properties.

- $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$
- $\frac{1}{\sigma^2} \sum_{i=1}^n (X_i - \bar{X})^2 \sim \chi^2_{(n-1)}$
- the r.v.s in the two parts immediately above are independent.
- If $Z \sim N(0,1)$ and $Y \sim \chi^2_{(m)}$ are independent then

$$T = \frac{Z}{\sqrt{Y/m}} \sim t_{(m)}$$

  that is the student's t distribution with m degrees of freedom.
- A $\chi^2_{(m)}$ distribution is also a Gamma$(\frac{m}{2}, \frac{1}{2})$ distribution.

Use these properties as needed to answer the following :

i) derive the distribution of

$$\frac{\sqrt{n}(\bar{X} - \mu)}{S}$$

where

$$S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2 \ .$$

ii) Use Chebyshev's inequality to prove that S^2 converges in probability to σ^2.
Hint : Recall Question 1 where you stated Chebyshev's inequality. Determine how to apply this inequality in this problem.

c) Suppose that X, Y are independent r.v.s and that $X \sim \chi^2_{(k)}$ and $Y \sim \chi^2_{(\ell)}$. Consider the r.v. $W = \frac{Y/\ell}{X/k}$.

Find the pdf of W.

5. Suppose that N is a random variable with a geometric, p, distribution, that is N has pmf (probability mass function)

$$P(N = n) = (1 - p)^{n-1} p \ , n = 1, 2, 3, \ldots$$

and $P(N = n) = 0$ for all other n.

Suppose that $X_i \sim \text{Bernoulli}(\theta)$ are iid, $i = 1, 2, \ldots$ and the X_i's are independent of N.

Let

$$S = \sum_{i=1}^{N} X_i \ .$$

a) *Derive* the moment generating function of the geometric distribution (above).

b) Find the formula $\mathrm{E}(e^{tS}|N = n)$ for integers $n \geq 1$. Give the conditional expectation $\mathrm{E}\left(e^{tS}|N\right)$. Using this find the MGF of S, say M_S. State which properties of conditional expectation you are using to obtain this.

c) Using the function M_S from (b) obtain $\mathrm{E}(S)$ and $\mathrm{E}(S^2)$.