University of Western Ontario
Statistics 3657 Term Test II

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Instructions:

I. Make sure that your name and ID number are the front page, and every page, of your question booklet.

II. This term test is of 50 minutes duration. Each question starts on a new page. Each question have approximately equal weights.

III. All questions are to be answered on the questions sheets. You may use the back of these pages, but please indicate where any answer continuation is located. In the absence of such directions work on the backs of pages will not be graded.

IV. An additional booklet will be provided, if needed, for your rough work. This will not be graded.

1	12
2	12
3	13
4	13
Total (50)	

Name : ______________________

ID : ______________

1. a) State the definition of Expected value for a discrete random variable.

see notes. Some condition should be stated such as $\sum_x x P(X=x)$ is well defined (recall not all sums or integrals are a number - see notes) or $\sum |x| P(X=x) < \infty$

b) One of the Theorems in this course is
Theorem Suppose that X, Y are jointly distributed discrete r.v.s and that $W = g(X, Y)$ for a function g. Then

$$\mathrm{E}(W) = \sum_x \sum_y g(x,y) P(X = x, Y = y) \ .$$

i) Prove this theorem.
ii) Discuss in one or two sentences why this theorem is useful.

(i) From (a) we have $E(W) = \sum_w w P(W=w)$.
The goal is to show this sum (over w) is also the same as the sum over x, y.
See notes.

c) Another Theorem in this course is
Theorem Suppose that X, Y are independent r.v.s. Let $W = g(X)h(Y)$ for function g, h. Under (conditions that you will give below) $\mathrm{E}(W) = \mathrm{E}(g(X))\mathrm{E}(h(Y))$.

i) State these conditions.
ii) Prove the theorem in the case of discrete r.v.s, indicating where you are using the conditions.

(i) Conditions are $E(|g(X)|)$, $E(|h(Y)|)$ are finite (may also write $\sum_x |g(x)| P(X=x) < \infty$ and $\sum_y |h(y)| P(Y=y) < \infty$.

(ii) The student may now use (b) (just proven above)

2. Suppose that $X \sim$ exponentialλ and that $Y|X = x \sim g_x$ (that is the conditional distribution of Y given that $X = x$ has pdf g_x) where the conditional pdf g_x is

$$g_x(y) = \lambda e^{-\lambda(y-x)} \mathrm{I}_{(x,\infty)}(y)$$

a) Are X and Y independent and explain your reason.

b) Find $E(Y|X)$, $E(Y^2|X)$, and $E(Y^2)$.

c) Find the marginal pdf of Y. By integration find $E(Y)$. Will this answer be the same as $E(E(Y|X))$ (YES or NO)? Justify your answer by stating an appropriate theorem if YES and by direct calculation if NO.

discussed in class on the blackboard.

3. Suppose that X, Y has pdf

$$f_{X,Y}(x,y) = c\frac{(x+y)^2}{x^2}\mathrm{I}_{(1,2)}(x)\mathrm{I}_{(1,2)}(y)$$

where $c = \frac{3}{4} + \log_e(2) = 1.443147$. You might find it best to just write c for this constant in your algebraic calculations, until you need to calculate a numerical answer such as for expectation.

Let

$$\begin{aligned} U &= X + Y \\ V &= \frac{Y}{X}. \end{aligned}$$

a) Find the joint pdf of U, V, and explicitly give the support of this pdf. Sketch this support.

b) Obtain the marginal distribution of U and use *this* to calculate the expected value of U.

discussed in class on the blackboard.

4. Suppose $X_1, X_2, \ldots, X_n$ are iid random variables with pdf f, and with cdf F. Let $X_{(k)}, X_{(\ell)}$ be the k-th and ℓ-th order statistics. Below the integers are such that $k < \ell$.

   a) State the pdf of $X_{(k)}, X_{(\ell)}$ (do not derive or prove this).

   b) Suppose X_i, $i = 1, 2, 3$ are iid random variables where the pdf f is given by

$$f(x) = \begin{cases} e^{-x} & \text{if } 0 \le x < \infty \\ 0 & \text{otherwise.} \end{cases}$$

   Aside : In this problem you are working with a sample of size $n = 3$.
   Give the pdf of $X_{(1)}, X_{(3)}$

   c) Continuation of the previous part. Calculate $\mathrm{E}(X_{(3)} - X_{(1)})$.

(a) Since some students interpreted this as giving the two marginal distributions, I accepted this answer as well.

(b)
$f_{1,3}$ = pdf of $X_{(1)}, X_{(3)}$.
We also need $F(x) = \int_{-\infty}^{x} f(y)\,dy = \begin{cases} 1 - e^{-x} & \text{if } x \ge 0 \\ 0 & \text{if } x < 0 \end{cases}$

$$f_{1,3}(x,y) = \begin{cases} \frac{3!}{1!\,1!\,1!} f(x)\,(F(y) - F(x)) \cdot f(y) & \text{if } x < y \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 6e^{-\lambda x}(e^{-x} - e^{-y}) \cdot e^{-y} & \text{if } 0 < x < y \\ 0 & \text{otherwise.} \end{cases}$$

(or a student could continue with interpretation in (a))

(c) There are two ways to do this part

(1) $E(X_{(3)} - X_{(1)}) = E(X_{(3)}) - E(X_{(1)})$ by linearity of expectation

Now the student will need the two marginal pdfs.

or (2) $E(X_{(3)} - X_{(1)}) = \int_0^\infty \int_x^\infty (y - x) f_{1,3}(x,y)\,dy\,dx$