University of Western Ontario
Statistics 3657 Term Test II

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Instructions:

I. Make sure that your name and ID number are the front page, and every page, of your question booklet.

II. This term test is of 50 minutes duration. Each question starts on a new page. Each question have approximately equal weights.

III. All questions are to be answered on the questions sheets. You may use the back of these pages, but please indicate where any answer continuation is located. In the absence of such directions work on the backs of pages will not be graded.

IV. An additional booklet will be provided, if needed, for your rough work. This will not be graded.

1	12
2	13
3	13
4	12
Total (50)	

3, 5, 4

Name : ____________________

ID : ______________

Brief Solutions

1. a) State the definition of Expected value for a discrete random variable.

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b) One of the Theorems in this course is

Theorem Suppose that X, Y are jointly distributed r.v.s and that $W = g(X, Y)$ for a function g. Then (with some assumptions - or conditions - that you will give)

5

$$\mathrm{E}(W) = \sum_x \sum_y g(x, y) P(X = x, Y = y) \ .$$

i) State these assumptions.

ii) Use this theorem, and state any additional assumptions that you will need, to show for constants a, b,

$$\mathrm{E}(aX + bY) = a\mathrm{E}(X) + b\mathrm{E}(Y) \ .$$

In particular give the function g that you will use.

c) Another Theorem in this course is

Theorem Suppose that X, Y are independent r.v.s. Let $W = g(X)h(Y)$ for function g, h. Under (conditions that you will give below) $\mathrm{E}(W) = \mathrm{E}(g(X))\mathrm{E}(h(Y))$.

i) State these conditions.
ii) Use this theorem, and state any additional assumptions that you require, to prove that for independent r.v.s X, Y, Z then

$$\mathrm{E}(XYZ) = \mathrm{E}(X)\mathrm{E}(Y)\mathrm{E}(Z) .$$

In your proof, if you are using any other theorems from earlier in the course, state these and indicate how or where you are using these in your proof.

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(i) see notes

(ii) XY and Z are independent
(state why).
By Theorem above, if X, Y have finite mean
then so does $\cancel{X}Y$, $E(XY) = E(X) \cdot E(Y)$
Now apply this theorem (above) to
XY and Z.
$\therefore E(XYZ) = E(XY) \cdot E(Z) = E(X)E(Y)E(Z)$

2. The r.v.s N and $X_i, i \geq 1$ are all independent, and the X_i are iid Poisson, 1, and N has pmf

$$P_N(n) = P(N = n) = \begin{cases} \frac{1}{4} & \text{if } n = 1 \\ \frac{1}{2} & \text{if } n = 2 \\ \frac{1}{4} & \text{if } n = 3 \\ 0 & \text{otherwise.} \end{cases}$$

Consider the r.v. (the random sum of r.v.s)

$$S = \sum_{i \geq 1}^{N} X_i \, .$$

a) Obtain (derive) the formula for $E(S|N = n)$ and $E(S^2|N = n)$.

b) Give the r.v.s $E(S|N)$, $E(S^2|N)$.

c) State an appropriate Theorem and use it to find $E(S)$, $E(S^2)$ and $Var(S)$.

d) Find $P(S = 1)$. *Hint : Use conditional probability. You may also use the property that a sum of m iid Poisson, λ, r.v.s has a Poisson, mλ, distribution.*

$E(N) = \frac{1}{4} + 2 \cdot \frac{1}{2} + 3 \cdot \frac{1}{4} = \frac{1+4+3}{4} = \frac{8}{4} = 2$

3 a) see notes, $E(S|N=n) = h_1(n) = E(X) \cdot n$

$E(S^2|N=n) = h_2(n) = E\left(\sum_{i=1}^{n}\sum_{j=1}^{n} X_i X_j\right) = n E(X^2) + n(n-1)E(X)^2$

$X_i \sim$ Poisson, 1 $\therefore E(X) = 1$, $Var(X) = 1$, $E(X^2) = Var(X) + E(X)^2 = 2$

3 (b) (c) $E(S) = E(E(S|N))$ (state appropriate theorem - see notes)

$= E(X) \cdot E(N) = 2$

4 $E(S^2) = E(X^2) \cdot E(N) + E(X)^2 \cdot E(N^2 - N)$

$= 2 + E(N^2) - E(N)$ $\qquad E(N^2) = \frac{1}{4} + \frac{4 \cdot 2}{4} + \frac{9}{4} = \frac{18}{4} = \frac{9}{2}$

$= 1 + \frac{9}{2}$

- finish to obtain $Var(S)$

(d) $P(S=1) = \sum_{n=1}^{3} P(S=1|N=n) \cdot P(N=n)$ $(S|N=n \sim \text{Poisson}, n)$

3 $= e^{-1} \cdot P(N=1) + 2 \cdot e^{-2} \cdot P(N=2) + 3 \cdot e^{-3} P(N=3)$

$= \frac{e^{-1}}{4} + 2 \cdot e^{-2} \cdot \frac{2}{4} + 3 \cdot e^{-3} \cdot \frac{1}{4}$

3. Suppose that X, Y are independent r.v.s., that $X \sim N(0,1)$ and that Y has pdf

$$f_Y(y) \quad = \quad e^{-y} I_{(0,\infty)}(y) \ .$$

Let

$$\begin{aligned} U &= \frac{X}{\sqrt{Y}} \\ V &= Y \ . \end{aligned}$$

4 a) Find the joint pdf of U, V, and explicitly give the support of this pdf. Sketch this support.

5 b) Obtain the marginal distribution of U.

4 c) Return now to the r.v. $\frac{X}{\sqrt{Y}}$. Using conditional expectation, find

$$h(y) = \mathrm{E}\left(\frac{X}{\sqrt{Y}} \mid Y = y\right) \ .$$

Give the r.v. $\mathrm{E}\left(\frac{X}{\sqrt{Y}} \mid Y\right)$, and then use this to obtain $E(U)$.

(c) Since X, Y are independent and $E(X) = 0$

then $E(X \mid Y = y) = E(X) = 0$

$\therefore E\left(\frac{X}{\sqrt{Y}} \mid Y\right) = \frac{1}{\sqrt{Y}} E(X \mid Y) = \frac{1}{\sqrt{Y}} h(Y) = 0$

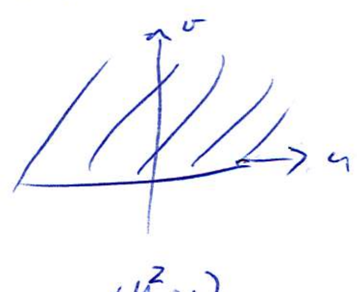
[(c) can be done without (b)]

(a) $$\frac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} \frac{\partial u\sqrt{v}}{\partial u} & \frac{\partial u\sqrt{v}}{\partial v} \\ \frac{\partial v}{\partial u} & \frac{\partial v}{\partial v} \end{pmatrix} = \begin{pmatrix} \sqrt{v} & \frac{u}{2\sqrt{v}} \\ 0 & 1 \end{pmatrix} = \sqrt{v}$$

$$\therefore f_{U,V}(u,v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(u\sqrt{v})^2} \cdot e^{-v} \sqrt{v}\, I(u \in \mathbb{R}, v > 0)$$

$$= \frac{1}{\sqrt{2\pi}} \sqrt{v}\, e^{-v\left(\frac{u^2}{2}+1\right)} I(u \in \mathbb{R}, v > 0)$$

Support



(b) $u \in \mathbb{R}$

$$f_U(u) = \int_0^\infty \frac{\sqrt{v}}{\sqrt{2\pi}} e^{-v\left(\frac{u^2}{2}+1\right)} dv$$

(change variables $w = v\left(\frac{u^2}{2}+1\right)$

or recognize this integrand is kernel of Gamma pdf)

$$= \frac{\Gamma\left(\frac{3}{2}\right)}{\sqrt{2\pi}} \cdot \left(\frac{u^2}{2}+1\right)^{-\frac{3}{2}}$$

You might also notice $\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$

so $f_U(u) = \frac{1}{2\sqrt{2}} \left(\frac{u^2}{2}+1\right)^{-\frac{3}{2}}$

4. Suppose $X_1, X_2, \ldots, X_n$ are iid random variables with pdf f, and with cdf F. Let $X_{(k)}$ be the k-th order statistic.

4 a) Using the heuristic method derive the pdf of $X_{(k)}$.

4 b) Suppose X_i, $i = 1, 2, 3, 4$ are iid random variables where the pdf f is given by

$$f(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Give the pdf of $X_{(3)}$ (that is the formula for this pdf).

4 c) Continuation of the previous part. Calculate $E(X_{(3)})$ and $E(X_{(3)}^2)$ (you can give these as an appropriate fraction or rational number, no need to give the decimal expression for these values).

(b) $F(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^2 & \text{if } 0 \le x \le 1 \\ 1 & \text{if } x > 1 \end{cases}$

$f_3 = \text{pdf of } X_{(3)}$

$f_3(x) = \frac{4!}{2!\,1!\,1!} \cdot F(x)^2 \cdot f(x) \,(1 - F(x))$

$= 12\, (x^2)^2 \cdot 2x\, (1 - x^2)\, I(0 \le x \le 1)$

$= 24\, x^5 (1 - x^2)\, I(0 \le x \le 1)$

$E(X_{(3)}) = 24\left(\frac{1}{7} - \frac{1}{9}\right) = \frac{48}{63} = \frac{16}{21}$

$E(X_{(3)}^2) = 24 \int_0^1 x^7 (1 - x^2)\, dx = 24\left(\frac{1}{8} - \frac{1}{10}\right) = \frac{6}{10}$