Ratio of Independent Standard Normal (Gaussian) RVs

Suppose that X and Y are independent r.v.'s each with $N(0,1)$ distribution. (We also that X, Y are iid $N(0,1)$. Consider the transformation of the r.v.'s

$$W = \frac{Y}{X}$$

Since X, Y are bivariate continuous (why?) then the pdf of W is

$$f_W(w) = f_W(w) = \int_{-\infty}^{\infty} |x|\phi(x)\phi(wx)dx$$

where ϕ is the standard normal pdf

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \ .$$

For this integrand the exponent is

$$-\frac{1}{2}\left(x^2 + w^2 x^2\right) \quad = \quad -\frac{(1+w^2)}{2}x^2$$

Thus

$$\begin{aligned}
f_W(w) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |x| \exp\left\{-\frac{(1+w^2)}{2}x^2\right\} dx \\
&= \frac{1}{\pi}\int_{0}^{\infty} |x| \exp\left\{-\frac{(1+w^2)}{2}x^2\right\} dx \quad (\text{ why can we do this?}) \\
& \quad \text{change variables } y = \frac{(1+w^2)}{2}x^2 \\
&= \frac{1}{\pi}\int_{0}^{\infty} \frac{\sqrt{2y}}{\sqrt{1+w^2}} \exp\{-y\} \frac{\sqrt{2}}{2\sqrt{y}\sqrt{1+w^2}} dy \\
&= \frac{1}{\pi}\int_{0}^{\infty} \frac{1}{\sqrt{1+w^2}} \exp\{-y\} \frac{1}{\sqrt{1+w^2}} dy \\
&= \frac{1}{\pi}\frac{1}{(1+w^2)}\int_{0}^{\infty} \exp\{-y\} dy \\
&= \frac{1}{\pi}\frac{1}{(1+w^2)}
\end{aligned}$$

This particular function is the pdf of the (standard) Cauchy distribution. Later we will see this distribution does not have an expectation, that the integral does not exist as a value in the reals or a = or + infinity. There is another way of obtaining a Cauchy r.v. besides the ratio of independent standard normals; see problem 2.5.39 in the Rice text.