Statistics 3657 : Some Notes on Taylor Series

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In our course we will make some use of Taylor's expansion, but only of order 1 and 2.

In these notes we consider only functions of either one or of 2 variables. These results are given in most introductory texts on the calculus of one or several variables.

Below derivatives of the necessary orders all exist for these functions.

Consider f on an appropriate domain which is a subset of the reals.

First order Taylor Approximation

$$f(x_0 + h) = f(x_0) + f'(x_0)h + R_1(x_0, h)$$

where $R_1(x_0, h)$ is the remainder term. The first order Taylor approximation to f is the first order polynomial with argument h

$$f_{[1]}(h) = f(x_0) + f'(x_0)h \ .$$

The remainder term is given by

$$R_1(x_0, h) = \frac{1}{2!} f''(x_0 + \theta h)h^2$$

for some number θ between 0 and 1. Typically the number θ cannot be easily calculated.

The notation with the subscript $[k]$ ($k = 1$ above) is nothing special. Here it is just a way of naming a polynomial that happens to be an approximation to f and I was running out of useful subscripts for the notation.

Second order Taylor Approximation

$$f(x_0 + h) = f(x_0) + f'(x_0)h + \frac{1}{2!} f''(x_0)h^2 + R_2(x_0, h)$$

where

$$R_2(x_0, h) = \frac{1}{3!} f^{(3)}(x_0 + \theta h)h^3$$

where θ is a number between 0 and 1. The second order Taylor approximation is the polynomial of degree 2, with argument h The first order Taylor approximation to f is the first order polynomial with argument h

$$f_{[2]}(h) = f(x_0) + f'(x_0)h + \frac{1}{2!} f''(x_0)h^2 \ .$$

The remainder terms for the n degree Taylor expression is written in terms of an $n + 1$-st derivative. We do not make explicit use of in this course but it is needed when one wants to demonstrate how accurate is a Taylor approximation.

Consider a real valued function of two variables (arguments) with appropriate domain a subset of $R^2 = R \times R$.

For notation for partial derivatives write

$$\frac{\partial^{n+m}}{\partial x^n \partial y^m} f(x,y) = f_{nm}(x,y) \ .$$

First order Taylor Approximation

$$f(x_0+h, y_0+k) = f(x_0,y_0) + f_{10}(x_0,y_0)h + f_{01}(x_0,y_0)k + R_1(x_0,y_0,h,k)$$

where $R_1(x_0,y_0,h,k)$ is the remainder term. The first order Taylor approximation to f is the first order polynomial with argument h,k

$$f_{[1]}(h,k) = f(x_0,y_0) + f_{10}(x_0,y_0)h + f_{01}(x_0,y_0)k \ .$$

The remainder term is given by

$$R_1(x_0,y_0,h,k) = \frac{1}{2!} f_{20}(x_0+\theta h, y_0+\theta k)h^2 + f_{11}(x_0+\theta h, y_0+\theta k)hk + \frac{1}{2!} f_{02}(x_0+\theta h, y_0+\theta k)k^2$$

for some number θ between 0 and 1. Typically the number θ cannot be easily calculated.

Second order Taylor Approximation The Taylor's approximation of order (degree 2) is a second degree polynomial

$$\begin{aligned} f_{[2]}(h,k) &= f(x_0,y_0) + f_{10}(x_0,y_0)h + f_{01}(x_0,y_0)k \\ &\quad + \frac{1}{2!} f_{20}(x_0,y_0)h^2 + f_{11}(x_0,y_0)hk + \frac{1}{2!} f_{02}(x_0,y_0)k^2 \end{aligned}$$

In these notes I will not explicitly give the remainder term. It is given in most advanced calculus texts, but apparently no the Stewart text for the second year advanced calculus course at UWO.