Bivariate and Multivariate Normal

1 Bivariate Normal

The bivariate normal pdf is given by

$$
\begin{aligned}
f(x,y) \quad = \quad & \tfrac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left\{ -\tfrac{1}{2(1-\rho^2)} \left[\left(\tfrac{x-\mu_X}{\sigma_X}\right)^2 + \left(\tfrac{y-\mu_Y}{\sigma_Y}\right)^2 \right.\right. \\
& \left.\left. -2\rho\left(\tfrac{x-\mu_X}{\sigma_X}\right)\left(\tfrac{y-\mu_Y}{\sigma_Y}\right)\right]\right\}
\end{aligned} \tag{1}
$$

It has 5 parameters $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho$.

There is another nice way of expressing this pdf. Let

$$
\mathbf{Q} = \begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix}
$$

Then

$$
f(x,y) = \frac{1}{2\pi\sqrt{\det(\mathbf{Q})}} \exp\left\{ -\frac{1}{2}\left(x-\mu_X, y-\mu_Y\right)\mathbf{Q}^{-1}\left(x-\mu_X, y-\mu_Y\right)^T \right\} \tag{2}
$$

(in the above expression the superscript T means the transpose of the row vector.) In this formulation (equation 2) the matrix $\mathbf{Q}$ is a symmetric positive definite 2 by 2 matrix.

This bivariate pdf can be generalized to a multivariate normal pdf. Let $\mathbf{x}$ and μ be k dimensional row vectors. $\mathbf{x}$ will be the argument of a function and μ will be a (vector) parameter. Let $\mathbf{Q}$ be a $k \times k$ positive definite matrix. Then the function $f : R^k \mapsto R$ given by

$$
f(\mathbf{x}) = \frac{1}{(2\pi)^{k/2}\sqrt{\det(\mathbf{Q})}} e^{-\frac{1}{2}(\mathbf{x}-\mu)\mathbf{Q}^{-1}(\mathbf{x}-\mu)^T}
$$

is called the k dimensional multivariate normal probability density function. Equation (2) is of this form with $k = 2$.

Marginal distribution of a bivariate normal.

Suppose X, Y has the bivariate distribution given above.

$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f(x,y)dy \\
&= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right.\right. \\
&\qquad \left.\left. -2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right)\right]\right\} dy \\
&= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right\} \\
&\quad \times \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[-2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]\right\} dy
\end{aligned}$$

Change variables $u = \frac{y-\mu_Y}{\sigma_Y}$ in this last integral gives

$$\begin{aligned}
&\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[-2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]\right\} dy \\
&= \sigma_Y \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[-2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)u + u^2\right]\right\} du \\
&\quad \text{(next complete the square in } u\text{)} \\
&= \sigma_Y \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[u - \left(\frac{x-\mu_X}{\sigma_X}\right)\right]^2\right\} du \times \exp\left\{\frac{\rho^2}{2(1-\rho^2)}\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right\} \\
&= \sigma_Y\sqrt{2\pi(1-\rho^2)} \exp\left\{\frac{\rho^2}{2(1-\rho^2)}\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right\}
\end{aligned}$$

Thus

$$\begin{aligned}
f_X(x) &= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right\} \\
&\quad \times \sigma_Y\sqrt{2\pi(1-\rho^2)} \exp\left\{\frac{\rho^2}{2(1-\rho^2)}\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right\} \\
&= \frac{1}{\sqrt{2\pi}\sigma_X} \exp\left\{-\frac{1-\rho^2}{2(1-\rho^2)}\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right\} \\
&= \frac{1}{\sqrt{2\pi}\sigma_X} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right\}
\end{aligned}$$

which is the $N(\mu_X, \sigma_X^2)$ density function.

By a symmetric argument we find

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx$$

$$= \frac{1}{\sqrt{2\pi}\sigma_Y} \exp\left\{ -\frac{1}{2} \left(\frac{y-\mu_Y}{\sigma_Y} \right)^2 \right\}$$

which is the $N(\mu_Y, \sigma_Y^2)$ density function.

Note that in general $f(x,y) \neq f_X(x) f_Y(y)$, so that X and Y are dependent.

Next consider the case $\rho = 0$. In this case we have $f(x,y) = f_X(x) f_Y(y)$, so that X and Y are independent. Thus in the normal case we X and Y are independent if and only if $\rho = 0$.

Later in the course we will see

$$\rho \sigma_X \sigma_Y = \mathrm{Cov}(X,Y) = \mathrm{E}\left((X-\mu_X)(Y-\mu_Y)\right)$$

Next the correlation of X with Y is

$$\mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y} = \rho$$

so that ρ is the correlation of X with Y.

Conditional distributions of a bivariate normal.

The conditional probability density function of Y given $X = x$ is defined for all $x \in R$ since the marginal distribution of X has support R. It is given by

$$\begin{aligned}
f_{Y|X=x}(y) &= \frac{f(x,y)}{f_X(x)} \\
&= \frac{\sqrt{2\pi}\sigma_X}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right.\right. \\
&\quad \left.\left. -2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right)\right] + \frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right\} \\
&= \frac{1}{\sigma_Y\sqrt{2\pi(1-\rho^2)}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 + \rho^2\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right.\right. \\
&\quad \left.\left. -2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right)\right]\right\} \\
&\quad \text{(now complete the square in } y\text{)} \\
&= \frac{1}{\sigma_Y\sqrt{2\pi(1-\rho^2)}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{y-\mu_Y}{\sigma_Y}\right) - \rho\left(\frac{x-\mu_X}{\sigma_X}\right)\right]^2\right\} \\
&= \frac{1}{\sigma_Y\sqrt{2\pi(1-\rho^2)}} \exp\left\{-\frac{1}{2\sigma_Y^2(1-\rho^2)}\left[y-\mu_Y-\rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X)\right]^2\right\}
\end{aligned}$$

which is the normal density with mean given by $\mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X)$ and variance given by $(1-\rho^2)\sigma_Y^2$.

In particular notice that the conditional mean of Y given X is a linear function of X.

A second thing to notice is that in the case $\rho = 0$ this conditional density is the $N(\mu_Y, \sigma_Y^2)$ density, the same as the marginal pdf of Y. Thus in the case that $\rho = 0$, we have independence of Y and X.

Correlation for bivariate normal

Consider the bivariate normal distribution (1) with parameters $(0, 0, 1, 1, \rho)$. Suppose that (X, Y) has this bivariate normal distribution. Then X and Y both have the standard normal distribution as their marginal distributions. In class we calculated the mean and variance of a $N(\mu, \sigma^2)$ distribution.

$$\mathrm{E}(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[x^2 - 2\rho xy + y^2\right]\right\} dydx$$

We can evaluate the inner integral

$$\begin{aligned}
A &= \int_{-\infty}^{\infty} y\frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[x^2 - 2\rho xy + y^2\right]\right\} dy \\
&= \int_{-\infty}^{\infty} y\frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[y^2 - 2\rho xy + \rho^2 x^2\right]\right\} dy\ e^{-\frac{x^2(1-\rho^2)}{2(1-\rho^2)}} \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} y\frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[(y-\rho x)^2\right]\right\} dy\ e^{-\frac{1}{2}x^2} \\
&= \frac{1}{\sqrt{2\pi}}\rho x e^{-\frac{1}{2}x^2}
\end{aligned}$$

since the integral is the mean of a $N(\rho x, (1-\rho^2))$ distribution.

Substituting this into the double (or iterated) integral we obtain

$$\begin{aligned}
\mathrm{E}(XY) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x\rho x e^{-\frac{1}{2}x^2} dx \\
&= \rho\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x^2 e^{-\frac{1}{2}x^2} dx \\
&= \rho \mathrm{Var}(X) \\
&= \rho
\end{aligned}$$

since marginally $X \sim N(0, 1)$.

Now suppose that (X, Y) has the bivariate normal distribution (1) of the general form. Consider the transform

$$\begin{aligned}
X_1 &= \frac{X-\mu_X}{\sigma_X} \\
Y_1 &= \frac{Y-\mu_Y}{\sigma_Y}
\end{aligned}$$

Then (X_1, Y_1) has the bivariate normal distribution with parameters $(0, 0, 1, 1, \rho)$. Thus $\mathrm{E}(X_1Y_1) = \rho$. Also

$$\mathrm{E}(X_1Y_1) = \mathrm{E}\left[\left(\frac{X-\mu_X}{\sigma_X}\right)\left(\frac{Y-\mu_Y}{\sigma_Y}\right)\right]$$

From the linearity property of expectation we then obtain

$$\begin{aligned}\mathrm{Cov}(X,Y) &= \mathrm{E}\left[(X-\mu_X)(Y-\mu_Y)\right] \\ &= \mathrm{E}(X_1Y_1)\sigma_X\sigma_Y \\ &= \rho\sigma_X\sigma_Y\end{aligned}$$

Finally we then obtain

$$\begin{aligned}\mathrm{Corr}(X,Y) &= \frac{\mathrm{Cov}(X,Y)}{\sigma_X\sigma_Y} \\ &= \rho\end{aligned}$$

2 Linear Transformations

There is a nice way of studying linear transformations of multivariate normal r.v.s using linear algebra.

Recall we can write the bivariate normal pdf as

$$f(x,y) = \frac{1}{(2\pi)^{k/2}\sqrt{\det(\mathbf{Q})}} e^{-\frac{1}{2}(x-\mu_X, y-\mu_Y)\mathbf{Q}^{-1}(x-\mu_X, y-\mu_Y)^T}$$

where

$$Q = \begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix}$$

Consider the transformation

$$\begin{pmatrix} U \\ V \end{pmatrix} = A \begin{pmatrix} X \\ Y \end{pmatrix} + B \tag{3}$$

where

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \text{ and } B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} .$$

and that A is invertible. In this case

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} a_{2,2} & -a_{2,1} \\ -a_{1,2} & a_{1,1} \end{pmatrix} .$$

The student should finish this example at home, that is find the joint pdf of U, V. The answer is that this joint distribution will be bivariate normal, but what are the parameters for this bivariate normal.

3 Sums of Normals : Bivariate normal case

In this section consider using the convolution formula to find the distribution of the sum of two r.v.s that have a bivariate normal distribution.

Suppose X, Y are bivariate normal(0,0,1,1,ρ), so they have density

$$f(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[x^2 - 2\rho xy + y^2\right]\right\}$$

Let $T = X + Y$. Then T has pdf f_T given by

$$f_T(t) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{\infty}^{\infty} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[x^2 - 2\rho x(t-x) + (t-x)^2\right]\right\} dx$$

How can we calculate this integral? First simplify the exponent.

$$\begin{aligned}
& x^2 - 2\rho x(t-x) + (t-x)^2 \\
&= x^2 + 2x(t-x) + (t-x)^2 - 2(1+\rho)x(t-x) \\
&= (x + (t-x))^2 + 2(1+\rho)x(x-t) \\
&= t^2 + 2(1+\rho)\left(x^2 - 2x\frac{t}{2} + \frac{t^2}{4} - \frac{t^2}{4}\right) \\
&= t^2 + 2(1+\rho)\left(x - \frac{t}{2}\right)^2 - 2(1+\rho)\frac{t^2}{4} \\
&= t^2\frac{1-\rho}{2} + 2(1+\rho)\left(x - \frac{t}{2}\right)^2
\end{aligned}$$

Therefore

$$\begin{aligned}
f_T(t) &= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{\infty}^{\infty} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[t^2\frac{1-\rho}{2} + 2(1+\rho)\left(x-\frac{t}{2}\right)^2\right]\right\} dx \\
&= \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{t^2\frac{1-\rho}{2}}{2(1-\rho)(1+\rho)}\right\} \\
&\quad \times \int_{\infty}^{\infty} \exp\left\{-\frac{1}{2(1-\rho)(1+\rho))}\left[\frac{1}{2}(1+\rho)\left(x-\frac{t}{2}\right)^2\right]\right\} dx \\
&= \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{t^2}{4(1+\rho)}\right\} \\
&\quad \times \int_{\infty}^{\infty} \exp\left\{-\frac{1}{2\frac{(1-\rho)}{2}}\left[\left(x-\frac{t}{2}\right)^2\right]\right\} dx \\
&= \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{t^2}{4(1+\rho)}\right\} \\
&\quad \times \sqrt{2\pi}\sqrt{\frac{(1-\rho)}{2}} \\
&= \frac{1}{\sqrt{2\pi(2(1+\rho))}} \exp\left\{-\frac{t^2}{4(1+\rho)}\right\}
\end{aligned}$$

Thus we see that T has the normal distribution parameters 0 and $2(1+\rho)$.

What happens for a more general bivariate normal, instead of this special case?