Statistics 3657a : Prerequisites Review Problems

Handout date September 4, 2014
The student should work through these problems.

Remarks : These question consists of review problems of types that are studied in calculus. A student may need to make use of appropriate texts in the library to review the material. It is advised that the student do so as needed. Going to the appropriate source text will help the student recall these various notation and definitions. All the problems come up later in the course, especially the parts related to calculus. The student should refer back to these at various places throughout the semester.

Questions

1. The Fundamental Theorem of Calculus is an important tool in some of our calculations.

   i: State the Fundamental Theorem of Calculus, with a reference to the text and page where you have located it. *It is worth a trip to the library.*

   ii: Suppose $f : R \mapsto R$ and $f(x) = e^{-|x|}$.

   a) Let $g_1(x) = \int_{-\infty}^{x} f(y)dy$ for $x \in R$. Find $g_1'(x) = \frac{dg_1(x)}{dx}$.
   Do this both by finding the formula for g_1 by integration (be careful with $x < 0$ and $x > 0$ cases) and by using the Fundamental Theorem of Calculus.
   Comment in one sentence if the Fundamental Theorem of Calculus makes this calculation easier or more difficult for you.
   Aside : You should notice that you can obtain the derivative by using the Fundamental Theorem of Calculus without having to find the formula for g_1 by solving the integral.

   b) Let $g_2(x) = \int_{-\infty}^{x^2} f(y)dy$ for $x \in R$. Find g_2'.

   c) Let

$$g_3(x) = \begin{cases} \int_{-\sqrt{x}}^{\sqrt{x}} f(y)dy & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find $g_3'(x)$ for $x \neq 0$. Note this also includes $x < 0$.
Show $g_3'(0)$ does not exist. *Hint* : Use the appropriate difference quotient and study its limit.

2. Ratio tests and comparison theorems are useful for determining when an integral produces a finite number. There is also a corresponding version for infinite sums.

   Also review the meaning of

   $$\int_{-\infty}^{\infty} f(x)dx$$

   for a given function. In particular the student should review how this integral is defined as a limit of a so called definite integral over a finite range, and should also review what it means for a sequence to have a defined limit and what it means for a sequence to have no defined limit. Some distributions have moments (expectations) of a given order and some do not, for example the Cauchy distribution does not have moments of any order or 1 or higher.

   i: Let $f(x) = e^{-x^2}$, $x \in R$.

   a) Prove that $\int_0^\infty f(x)dx$ is finite.
   Hint: Use a comparison theorem or ratio test to show this. Indicate how you are using the method for this problem.

   b) Prove that $A = \int_{-\infty}^\infty xf(x)dx = 0$.
   Hint : Show that $A^- = \int_{-\infty}^0 xf(x)dx$ is finite and that $A_+ = \int_0^\infty xf(x)dx$ is finite. Then use the fact that $A = A_- + A_+$ and show that $A_- = -A_+$.

   ii: Let $g(x) = \left(1 + x^2\right)^{-1}$ for $x \in R$. Is $A = \int_{-\infty}^\infty xg(x)dx$ equal to 0, some other finite number or undefined? Explain your answer in one sentence.
   Is $\int_{-\infty}^\infty g(x)dx$ a finite number? *You do not need to calculate this value, just determine if it is finite.*

3. In this problem you will consider functions $f : D \mapsto E$ for some domain and some range. The range E will be some subset of the real numbers, that is we only consider in this problem real valued functions. Also the domains D will be a subset of the real numbers, except the last part.

   i: Define monotone function and strictly monotone function. Give the name of the text and page number where you have found these definitions.

   ii: Give an example of a function which is strictly monotone increasing, and another function which is strictly monotone decreasing.

iii: Consider a function f which maps $x \mapsto x^2$. Give an example of such a function, including its domain and range, so this function is (i) strictly monotone and (ii) is not monotone.

iv: Define and give an example (this means you will need to include or specify the domain and range) of each

a) Onto function
b) Into function
c) (i) 1 to 1 function, and (ii) 1 to 1 and onto function
d) Now consider the domain and range D, E to be subsets of R^2. Give an example of a 1 to 1 mapping and an example of a not 1 to 1 mapping (also called a many to one mapping).

4. In many change of variables problems and integration one needs to consider a transformation from a pair of variables to another pair of variables. Consider

$$A = \{(x, y) : x \geq 0, y \geq 0\}$$

Consider the transformation

$$\begin{aligned} u &= x + y \\ v &= x - y \end{aligned}$$

Let B be the image of A under this transformation. Find B. Make two sketches, one of the region A and another of the region B. Shade the regions so that the marker is able to determine which are the appropriate regions you intend.

5. **Remark** : This problem will give you some practice and understanding calculations related to the following :

i: the inner integral in a multiple integral
ii: calculating a cumulative cdf from a pdf
iii: calculating a convolution formula

For these types of calculations one will need to find an expression where one considers x as *fixed* and y as the variable of integration. It is often the case that the limits of integration will depend upon (or change according to) the fixed or particular value of x. Another way of viewing this calculation is to produce a function with argument x. Below you will do such a calculation.

Consider the function

$$f(x) = \begin{cases} \frac{1}{2} & \text{if } 0 \leq x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Define the new function

$$g(x) = \int_{-\infty}^{\infty} f(y) f(x - y) dy \ . \tag{1}$$

Notice this is a function in which the domain is $R = (-\infty, \infty)$, the set of real numbers. Thus to give the rule or functional form for the function g you will need to a formula for $g(x)$ for all real numbers x. The limits of integration in (1) depend on x. It is possible for some x one integrates over an empty set.

Give the formula for $g(x)$ for all $x \in R$.
Hint: Be careful with the region of integration. You will obtain a different limits of integration and a formula for different regions of x. *Hint* : Sketch pictures of the support of the integrand in 1, and note how this support changes for different values of x. In particular there are some values of x for which the integral is calculated over an empty set.

6. **Remark** : This problem is related to calculating

   i: the inner integral in a multiple integral. Note that a double integral is

   $$\int_R \int_R f(x,y)dydx \quad = \quad \int_R \left\{ \int_R f(x,y)dy \right\} dx$$

   The double integral requires the evaluation of the (iterated) inner integral with respect to y to be calculated first. Note that the value of the inner integral is a function of x. You will find such functions in this problem.

   ii: calculating a cumulative cdf from a pdf and calculation of marginal distributions

   Suppose that f is a function of two real variables given by

   $$f(x,y) = \begin{cases} x & \text{if } 0 < x < 1 \text{ and } 0 < y < x \\ 0 & \text{otherwise} \end{cases}$$

   i: Sketch the support of f, that is the region $\{(x,y) : f(x,y) > 0\}$. More generally the support is $\{(x,y) : f(x,y) \neq 0\}$, but since $f(x,y) \geq 0$ in our case the support simplifies to the above.

   ii: Find $h(x) = \int_R f(x,y)dy$.
   Be careful since the answer will depend on whether x is less than 0, between 0 and 1, or bigger than 1.

   Aside : the algebraic expression you obtain will be different over certain subregions of the domain of h.

   iii: Find $\int_R f(x,y)dx$.
   Be careful, taking into account the region where y falls.

7. Keeping track of regions when making transformations, or change of variables in multiple integration is important in advanced calculus. Consider the region

   $$A = \{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$$

Make the transformation $u = x + y, v = x$. Sketch the region or image of this transformation.

Find $\int\int_{A\cap D} dydx$ where D is the region

$$D = \left\{(x, y) : x + y \leq \frac{1}{4}\right\}.$$

8. problem 2.5.51. This calculation determines the normalizing constant for the standard normal distribution.

9. Taylor's approximations are also quite useful in statistics and probability.

   i: Consider $f(x) = \log(x)$. Find the Taylor's approximation about $x_0 = 1$, of degree 1, say $f_1(x)$, and of degree 2, say $f_2(x)$.
   Sketch a plot of $f(x), f_1(x)$ and $f_2(x)$ on the region $[.5, 2]$. Briefly comment on the accuracy of the two approximations.

   ii: *The types of calculations below are useful in Chapter 5 related to the Central Limit Theorem.*
   n is an integer (related to sample size later). θ is a arbitrary fixed number.

   Suppose
   $$f(x) = e^{2(x-1)} - 1$$

   a) Calculate $f(1)$. Also find f'. Is it continuous at $x = 1$?
   b) Find the first order Taylor's approximation, about $x_0 = 1$, with error term or remainder term. Aside : This term will involve the second derivative of f. Call this first order Taylor's approximation f_1, and notice that it is a polynomial of degree 1.
   c) Use this first order Taylor's polynomial (that is Taylor's formula but without the remainder term) to give an approximate expression for
   $$f(1 + \frac{\theta}{n}) \ .$$
   Specifically this means to use the Taylor's formula f_1 but substitute in an appropriate place for x in this formula.
   Aside : In the terminology of an introductory calculus class, you are finding an expression for $f(1 + \Delta x)$ for small Δx, and will then treat for large n the term $\frac{\theta}{n}$ as Δx. This form allows one to find certain limits in a relatively easy way (next part of this question). This is how the Rice text studies the limit of a moment generating function for the standardized sample mean in Chapter 5.
   d) Given the expression for $nf(1 + \frac{\theta}{n})$ and then use part ii) a) to find
   $$\lim_{n\to\infty} nf(1 + \frac{\theta}{n})$$

e) State L'Hôpital's Theorem (may be called L'Hôpital's rule in some texts). Give the reference and page number of a text where you have found this.

f) Use it to calculate the limit

$$\lim_{y \to 0} \frac{f(1 + y\theta)}{y}$$

g) Can you use L'Hôpital's Theorem to calculate ii)d)? Hint : What happens if you can somehow replace $\frac{1}{n}$ by y? Discuss this.

iii: Consider the function

$$f(x, y) = \frac{x^2}{y - x^2} \ .$$

Give the Taylor's approximation of degree 1 to f, about the point $(x_0, y_0) = (1, 2)$.

Give the Taylor's approximation of degree 2.