Notes and Summary of Walter Rudin's REAL & COMPLEX ANALYSIS

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Contents

Introduction

What follows is a summary of the various chapters in Rudin's REAL & COMPLEX ANALYSIS. I developed these notes while studying for a qualifying exam in Analysis. From each chapter I have taken the theorems and definitions I felt deserved the most attention during my studies. This is not to say that these are the most important theorems or definitions; simply the ones which I chose to spend more time studying.

My goal here was to condense the enormity of Rudin's text to a digestible summary. Hence, there currently is very little in the way of comments or explanation of concepts. These can be found in the original text. I therefore emphasize that this work is not meant to be a substitute for the original text, but rather a complement.

I have, however, taken the liberty to include examples; one thing which is sorely lacking in the text. The examples I have included are from the preliminary exams I studied, and the solutions are my own. I make no guarantee that the solutions are correct, and suspect that there may be holes here and there. If you find mistakes in the solutions, or if you have any suggestions or comments, you can email me at

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You may notice that Chapter 7 on Differentiation is currently blank. This chapter was not covered during the course, and most of the information in it was familiar. I hope to add it later.

I have the files for this document archived at

http://www.math.utah.edu/~bobby/rudin.zip

The LaTeX files are included, as are all necessary .eps files. In this way, you are free to modify these notes toward your own studies. You should be aware, however, that several packages are required to compile the LaTeX files. In particular, the mathrsfs package is needed for the math script fonts.

Chapter 1

Abstract Integration

Definition. A collection $\mathfrak{M}$ of subsets of a set X is called a σ-algebra in X if $\mathfrak{M}$ has the following properties:

1. $X \in \mathfrak{M}$.
2. $A \in \mathfrak{M}$ implies $A^{\complement} \in \mathfrak{M}$.
3. If $\mathscr{A}$ is a countable collection of sets in $\mathfrak{M}$, then $\bigcup \mathscr{A} \in \mathfrak{M}$.

If $\mathfrak{M}$ is a σ-algebra in X, then X is called a measurable space, and the sets in $\mathfrak{M}$ are called the measurable sets in X.

If X is a measurable space, and Y is a topological space, and $f : X \longrightarrow Y$, then f is said to be a measurable function provided that $f^{-1}(V)$ is a measurable set in X for every open set V in Y.

Theorem. Let $g : Y \longrightarrow Z$ be a continuous map between topological spaces. If X is a measurable space and $f : X \longrightarrow Y$ is a measurable function, then $h = g \circ f$ is measurable.

Example 1. Let f and g be real-valued measurable functions on a measurable space X. Show that

$$E = \{x \in X \, : \, \sin f(x) \geq \cos g(x)\}$$

is a measurable subset of X.

Proof. Let $h = \sin \circ f - \cos \circ g$. Then, by continuity of sin and cos, and the measurability of f and g, we may conclude that h is measurable. Furthermore, $E = h^{-1}(-\infty, 0)^{\complement}$, and $(-\infty, 0)$ is an open set. Therefore the inverse image under h of $(-\infty, 0)$ is measurable, and the complement of such is also measurable. $\square$

Theorem (Lebesgue Monotone Convergence Theorem). Let $\{f_n\}$ be a sequence of measurable functions on X such that

1. $0 \le f_1(x) \le f_2(x) \le \ldots \le \infty, \quad \forall x \in X,$
2. $f_n(x) \to f(x)$ as $n \to \infty, \quad \forall x \in X.$

Then f is measurable, and

$$\int_X f_n \, d\mu \to \int_X f \, d\mu \quad \text{as} \quad n \to \infty.$$

Theorem. Let $f_n : X \longrightarrow [0, \infty]$ be measurable for $n \in \mathbb{N}$, and

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad \forall x \in X.$$

Then

$$\int_X f \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu.$$

Proof. Apply LMCT twice. □

Theorem (Fatou's Lemma). Let $f_n : X \longrightarrow [0, \infty]$ be measurable for $n \in \mathbb{N}$, then

$$\int_X \left(\liminf_{n\to\infty} f_n \right) d\mu \le \liminf_{n\to\infty} \int_X f_n \, d\mu$$

Definition. Let $\mathfrak{M}$ be a σ-algebra of X and let μ be a positive measure on $\mathfrak{M}$. We define the space of functions $L^1(\mu)$ via

$$L^1(\mu) = \left\{ f : X \longrightarrow \mathbb{C} \cup \{\infty\} : \int_X |f| \, d\mu < \infty \right\}.$$

If $f \in L^1(\mu)$, we say f is integrable.

Theorem. If $f \in L^1(\mu)$, then

$$\left| \int_X f \, d\mu \right| \le \int_X |f| \, d\mu.$$

Theorem (Lebesgue Dominated Convergence Theorem). Let $\{f_n\}$ be a sequence of measurable functions on X such that

$$f(x) = \lim_{n\to\infty} f_n(x)$$

exists for every $x \in X$. If there is a $g \in L^1(\mu)$ with

$$|f_n(x)| \leq g(x) \quad \forall n \in \mathbb{N}, \forall x \in X,$$

then $f \in L^1(\mu)$,

$$\lim_{n\to\infty} \int_X |f_n - f| \, d\mu = 0,$$

and

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

Proof. Uses Fatou's Lemma □

Example 2. Let $(X, \mathfrak{M}, \mu)$ be a positive measure space, and let $f \in L^1(\mu)$. Let $X_n = \{x \in X : |f(x)| > n\}$. Show that

$$\lim_{n\to\infty} n\,\mu(X_n) = 0.$$

Proof. Let $\chi_n = \chi_{X_n}$ be the characteristic function, and define $f_n = |f \cdot \chi_n|$. For every $x \in X$ there exists $M \in \mathbb{N}$ such that $|f(x)| < M$. [1] This implies that for all $n > M$, $\chi_n(x) = 0$. Therefore, $f_n(x) \to 0$ for all $x \in X$. Notice that

$$n\,\mu(X_n) < \int_{X_n} |f| \, d\mu = \int_X f_n \, d\mu.$$

Furthermore f_n are all dominated by f. Thus, by applying Lebesgue dominated convergence, we conclude that

$$\lim_{n\to\infty} n\,\mu(X_n) \leq \lim_{n\to\infty} \int_X f_n \, d\mu \leq \int_X \lim_{n\to\infty} f_n \, d\mu = 0.$$

The result follows. □

[1] There is the following subtlety here: it is actually possible that no $M \in \mathbb{N}$ bounds $|f(x)|$ since $f(x)$ could possibly be ∞. However, the set on which that can happen must have measure zero since $f \in L^1(\mu)$. Thus, we can replace f with another L^1 function by forcing those values to be zero, without loss of generality.

Example 3. Let μ be a positive measure on a measure space $(X, \mathfrak{M})$, and let f be a complex valued function on X which is integrable with respect to μ. For each n, set

$$E_n = \{x \in X \,:\, 1/n \leq |f(x)| \leq n\}.$$

Then each E_n is a measurable set on which $|f|$ is bounded. Prove that

1. $\mu(E_n)$ is finite for each n, and
2. $\displaystyle\lim_{n\to\infty} \int_{E_n} f \, d\mu = \int_X f \, d\mu.$

Proof. Let $f_n = f\chi_{E_n}$. Let $x \in X$. If $f(x) = 0$, then $f_n(x) = 0$ for all $n \in \mathbb{N}$. If $|f(x)| = \varepsilon > 0$ then there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\frac{1}{n} < \varepsilon < n.$$

Thus $f_n(x) = f(x)$ for all $n \geq N$. We conclude that $f_n \to f$, and $|f_n| \leq |f|$. Thus, we apply Lebesgue dominated convergence theorem to get

1. Since $f \in L^1(\mu)$,

$$n\mu(E_n) = \int_{E_n} n \, d\mu \leq \int_X |f_n| \, d\mu \leq \int_X |f| \, d\mu < \infty.$$

   Therefore, $\mu(E_n)$ is finite.

2. $$\lim_{n\to\infty} \int_{E_n} f \, d\mu = \lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

□

Chapter 2

Positive Borel Measures

Definition. If $\{x : f(x) > \alpha\}$ is open for all real α, then f is lower semicontinuous.
If $\{x : f(x) < \alpha\}$ is open for all real α, then f is upper semicontinuous.

Theorem (Urysohn's Lemma (Partitions of Unity)). Let X be a locally compact hausdorff space, V an open set in X, and K a compact subset of V. Then there exists a function $f \in C_c(X)$ (the space of continuous functions on X with compact support) with

$$0 \leq f \leq 1, \quad f\big|_K \equiv 1, \quad f\big|_{X \smallsetminus V} \equiv 0.$$

(The last condition is equivalent to support$(f) \subseteq V$).

Theorem (Riesz Representation Theorem). Let X be a locally compact hausdorff space, λ a positive linear functional on $C_c(X)$. Then there exists a σ-algebra $\mathfrak{M}$ in X and a measure μ on $\mathfrak{M}$ which "represents" λ:

1. $\lambda f = \int_X f \, d\mu \quad \forall f \in C_c(X)$,
2. $\mu(K) < \infty$ whenever K is compact,
3. $\forall E \in \mathfrak{M}$,

   $$\mu(E) = \inf \left\{ \mu(V) \middle| V \text{ is open and contains } E \right\}.$$

   (μ is outer-regular).
4. If E is open, or if $\mu(E) < \infty$, then

   $$\mu(E) = \sup \left\{ \mu(K) \middle| K \text{ is a compact subset of } E \right\}.$$

   (μ is inner-regular).
5. If $E \in \mathfrak{M}, A \subseteq E, \mu(E) = 0$, then $A \in \mathfrak{M}$.

Chapter 3

L^p-Spaces

Definition. $L^p(\mu) = \left\{ f : X \to \mathbb{C} \cup \{\infty\} : \left(\int_X |f|^p \, d\mu \right)^{1/p} < \infty \right\}.$

Theorem. Let p and q be conjugate exponents ($\frac{1}{p} + \frac{1}{q} = 1$), $1 < p < \infty$. Let X be a measure space with measure μ. Let f and g be measurable functions on X with range in $[0, \infty]$. Then the following inequalities hold:

- (Hölder's Inequality).

$$\int_X fg \, d\mu \le \left(\int_X f^p \, d\mu \right)^{1/p} \left(\int_X g^q \, d\mu \right)^{1/q}.$$

- (Schwarz Inequality). Take $p = q = 2$ above.
- (Minkowski Inequality).

$$\left(\int_X (f+g)^p \, d\mu \right)^{1/p} \le \left(\int_X f^p \, d\mu \right)^{1/p} + \left(\int_X g^p \, d\mu \right)^{1/p}.$$

This is the triangle inequality for $L^p(\mu)$: $\|f + g\|_p \le \|f\|_p + \|g\|_p$.

Definition ($\|f\|_\infty$). Define the essential supremum of f, denoted $\|f\|_\infty$, by the property that

$$|f(x)| \le \alpha \text{ for almost all } x \quad \iff \quad \|f\|_\infty \le \alpha.$$

Theorem (this is a summary of a few theorems). Let p and q be conjugate exponents, $1 \leq p \leq \infty$, and $f, g \in L^p(\mu)$, and $h \in L^q(\mu)$. Then:

1. $fh \in L^1(\mu)$ with $\|fh\|_1 \leq \|f\|_p \|h\|_q$,
2. $f + g \in L^p(\mu)$ with $\|f + g\|_1 \leq \|f\|_p + \|g\|_p$,
3. If $\alpha \in \mathbb{C}$ then $\alpha f \in L^p(\mu)$ with $\|\alpha f\|_p = |\alpha| \, \|f\|_p$,
4. $L^p(\mu)$ is complete (a Banach space).
5. If $p < \infty$ then the simple functions are dense in $L^p(\mu)$.
6. If $p < \infty$ then $C_c(X)$ is dense in $L^p(\mu)$, when X is a locally compact Hausdorff space and μ is regular.

Example 4. Let $0 < p < r < q$. Show that if $(X, \mathfrak{M}, \mu)$ is a (positive) measure space, then

$$L^r(\mu) \subseteq L^p(\mu) + L^q(\mu).$$

Proof. Let $f \in L^r(\mu)$. Then $|f|^r \in L^1(\mu)$. Let

$$A = \{x \in X \, : \, |f(x)| > 1\}$$

$$B = \{x \in X \, : \, |f(x)| \leq 1\}.$$

Define $g = f\chi_A$ and $h = f\chi_B$. Then $f = g + h$ and since $0 < p < r < q$, $|g|^p, |h|^q \leq |f|^r$ on X. Therefore, $f \in L^r(\mu)$ implies $g \in L^p(\mu)$ and $h \in L^q(\mu)$. □

Example 5. Let X be a measure space with $\mu(X) =!$. Let f be a measureable function in $L^\infty(X)$. Not that $\|f\|_p \leq \|f\|_\infty$ for all $p \geq 1$. In particular, f can be considerd an element of $L^p(X)$ for all $p \geq 1$. Show that

$$\lim_{p \to \infty} \|f\|_p = \|f\|_\infty.$$

Proof. Except on a set of measure 0, $|f| \leq \|f\|_\infty$. Therefore,

$$\|f\|_p^p \leq \int_X \|f\|_\infty \, d\mu = \|f\|_\infty \mu(X) = \|f\|_\infty.$$

So $\|f\|_p$ is bounded above by $\|f\|_\infty$. We will use Höler's inequality to show that it is also non-decreasing. Let $r < s$, and let q be the conjugate exponent to $p = s/r$. Then consider

$$\int_X |1 \cdot f|^r \, d\mu \leq \left[\int_X |f|^{rp} \, d\mu \right]^{1/p} \left[\int_X 1^q \, d\mu \right]^{1/q}.$$

By the Hölder inequality. But notice that $rp = s$. So

$$\|f\|_r^r \leq \|f\|_s^r \cdot \mu(X)^{1/q}.$$

But $\mu(X) = 1$, so we take the rth root of both sides to get

$$\|f\|_r \leq \|f\|_s.$$

Hence, the p-norms are increasing, and bounded above by $\|f\|_\infty$. Therefore, the limit exists.

Now we claim that $\|f\|_\infty = \sup_p \|f\|_p$. If this is indeed the case, then the result follows. We already know it is an upper bound, so it suffices to show that it is the least upper bound. Suppose $0 < M < \|f\|_\infty$. Then there is some set A with $0 < \mu(A) \leq 1$ on which $|f| > M$. Then

$$M^p \mu(A) < \int_A |f|^p \, d\mu \leq \|f\|_p^p.$$

In other words, $M \sqrt[p]{\mu(A)} \leq \|f\|_p$. Since $\mu(A) \leq 1$, as $p \to \infty$, $\sqrt[p]{\mu(A)} \to 1$. Thus, there must be some p_0 such that if $r > p_0$, then $M < \|f\|_r$. Thus, $\|f\|_\infty$ is the least upper bound. $\square$

Example 6. Let $f(x) = 1/\sqrt{x}$ if $0 < x < 1$, $f(x) = 0$ for $x \notin (0,1)$. Let $\{r_n\}_{n=1}^\infty$ be an enumeration of the rationals. Show that if

$$g(x) = \sum_{n=1}^\infty \frac{f(x - r_n)}{2^n}$$

then $g \in L^1(\mathbb{R})$ and hence $g(x) < \infty$ a.e. Show however, that even though $g^2(x) < \infty$ a.e., that $g \notin L^2(\mathbb{R})$.

Proof. We first note that from ordinary calculus, $\int_{\mathbb{R}} f = \int_0^1 f = 2$, while $\int_{\mathbb{R}} f^2 = \infty$. Thus, $f \in L^1(\mathbb{R})$ but not $L^2(\mathbb{R})$. Now by a corollary to Lebesgue Monotone Convergence, we may compute

$$\int_{\mathbb{R}} g(x) \, dx = \sum_{n=1}^\infty \frac{1}{2^n} \int_{\mathbb{R}} f(x - r_n) \, dx.$$

By translation invariance of the integral, the righthand side becomes $\sum \frac{1}{2^{n-1}}$, which is finite. Thus, $g \in L^1(\mu)$, and hence $g(x) < \infty$ a.e.

Since $g(x) < \infty$ a.e., $g^2(x) < \infty$ a.e. But

$$\int_{\mathbb{R}} [g(x)]^2 \, dx \geq \sum_{n=1}^\infty \frac{1}{2^{2n}} \int_{\mathbb{R}} [f(x - r_n)]^2 \, dx$$

by ignoring the cross terms in the square of the series. But now the righthand side is ∞. Thus, $g \notin L^2(\mu)$. $\square$

Chapter 4

Elementary Hilbert Space Theory

Definition (Inner Product Space). A complex vector space H is called an inner product space if for each $x, y \in H$ there exists a complex number $\langle x, y \rangle$, the so-called inner product of x and y, so that the following hold:

1. $\langle y, x \rangle = \overline{\langle x, y \rangle}$. (The inner product is skew symmetric or Hermitian)
2. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ for all $x, y, z \in H$.
3. $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$ for all $x, y \in H$ and $\alpha \in \mathbb{C}$.
4. $\langle x, x \rangle \geq 0$ for all $x \in H$.
5. $\langle x, x \rangle = 0 \iff x = 0$.
6. We can define the norm, $\|x\|$ of $x \in H$ by

$$\|x\|^2 = \langle x, x \rangle.$$

Theorem (Schwarz Inequality). $|\langle x, y \rangle| \leq \|x\| \, \|y\|$ for all $x, y \in H$

Theorem (Triangle Inequality). $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in H$.

Definition (Hilbert Space). By the Triangle inequality, we can define a metric on any inner product space H. If H is complete in this metric (all Cauchy sequences converge in H) then we say that H is a Hilbert space.

Lemma (Parallelogram Law). If $x, y \in H$, an inner product space, then

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Essentially, this says that if we have a parallelogram formed from the vectors x and y, then the sum of the squares of the diagonals is equal to the sum of the squares of the sides. See Figure 4.1

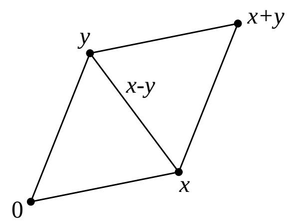

Figure 4.1: The parallelogram law

Proof. Write down $\|x+y\|^2 = \langle x+y, x+y\rangle$ and $\|x-y\|^2 = \langle x-y, x-y\rangle$. Simplify according to the definition of inner product and the result follows. $\square$

Example 7. Let $\mathbb{C}^n = \{z = (z_1, \ldots, z_n) : z_i \in \mathbb{C}\}$ be the n-dimensional complex vector space equipped with the norm

$$\|z\| = \sum_{i=1}^{n} |z_i|.$$

Prove that this norm does not come from inner product. That is, show there is no inner product on $\mathbb{C}^n$ with the property that $\|z\|^2 = \langle z, z\rangle$, for all $z \in \mathbb{C}^n$.

Proof. If $\|\cdot\|$ does come from inner product, then the parallelogram law must hold for all $x, y \in \mathbb{C}^n$. Let x be such that $x_1 = 1$ and all other entries are zero. Let y be such that $y_2 = 1$ and all other entries are zero. Then

$$\|x\| = \|y\| = 1,$$

and

$$\|x+y\| = \|x-y\| = 2.$$

Thus the parallelogram law fails since $8 \neq 4$. $\square$

Example 8. Show that the parallelogram law fails for $L^1([0,1])$ so it is not a Hilbert space.

Proof. Let $f = \chi_{[0,1/2]}$ and $g = \chi_{[1/2,1]}$. Then $\|f\|_1 = \|g\|_1 = \frac{1}{2}$, and $\|f+g\|_1 = \|f-g\|_1 = 1$. But

$$1^2 + 1^2 = 2 \neq 1 = 2\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right)^2.$$

$\square$

Here is another classification of Hilbert spaces:

Theorem. Every nonempty closed convex set in a Hilbert space has a **unique** element of minimal norm.

Example 9. Here is another way to do Example 7. Consider $n = 2$ (the higher dimensions follow). Then consider the unit ball $B = \{z : \|z\| \leq 1$. Translate B by the vector $w = (1, 1)$. Then $B + w$ has no unique element of minimal norm.

Theorem. If H is a Hilbert space, and $y \in H$ is fixed, then the mappings

$$x \mapsto \langle x, y \rangle, \quad x \mapsto \langle y, x \rangle, \quad x \mapsto \|x\|$$

are all continuous.

Orthogonality

Definition. Let $x, y \in H$. If $\langle x, y \rangle = 0$, we say x and y are orthogonal and write $x \perp y$. Let $x^\perp$ denote the subspace of all y which are orthogonal to x. If M is a subspace of H, let $M^\perp$ denote the space of all $y \in H$ which are orthogonal to every $x \in M$. Note that $x^\perp$ and $M^\perp$ are closed subspaces.

Theorem. Let M be a closed subspace of a Hilbert space H. Then the following hold:

1. Every $x \in H$ has a unique decomposition:

   $$x = Px + Qx$$

   into a sum of $Px \in M$ and $Qx \in M^\perp$.

2. Px and Qx are the nearest points to x in M and $M^\perp$, respectively.

3. The mappings $P : H \longrightarrow M$ and $Q : H \longrightarrow M^\perp$ are linear.

4. $\|x\|^2 = \|Px\|^2 + \|Qx\|^2$.

Corollary. If $M \neq H$ then there exists $y \in H$, $y \neq 0$, such that $y \perp M$.

Example 10. Let H be a Hilbert space, M a closed subspace, and N a finite-dimensional subspace of H. Show that $M + N$ is a closed subspace of H.

Proof. By induction, we may assume that N has dimension one. Thus $N = \{\lambda x_0 : \lambda \in \mathbb{C}\}$. If $x_0 \in M$ then we are done, because $M + N = M$ in this case. If $x_0 \notin M$ then we may write $x_0 = x_1 + x_2$ where $x_1 \in M$ and $x_2 \in M^\perp$. So without loss of generality, we may assume that $x_1 = 0$, and thus $N \perp M$.

Now let $(m_n + \lambda_n x_0)$ be a Cauchy sequence in H converging to x. We will show that $x \in M + N$. Let $P : H \to M$ and $Q : H \to M^\perp$ be orthogonal projections. Then $(m_n + \lambda_n x_0) \to x$ implies that $P(m_n + \lambda_n x_0) \to Px$ and $Q(m_n + \lambda_n x_0) \to Qx$. Therefore, $m_n \to Px$ and $\lambda_n x_0 \to Qx$. But this means that λ_n converges to some $\lambda \in \mathbb{C}$. Hence, we conlcude that $x = Px + \lambda x_0$, which is in $M + N$. $\square$

Example 11. Let P be a self-adjoint operator on a Hilbert space H such that $P^2 = P$. Show that P is a projection on a closed subspace. (By self-adjoint we mean that $\langle Px, y\rangle = \langle x, Py\rangle$ for all $x, y \in H$).

Proof. We have $P : H \longrightarrow H$. Let $M = \mathrm{img} P$, and $K = \ker P$. If we can show that $M = K^\perp$, then it will be closed. First, we note that

$$M^\perp = \{x \in H \,:\, \langle x, Py\rangle = 0, \text{ for all } y \in H\}.$$

Since P is self-adjoint, we may say

$$M^\perp = \{x \in H \,:\, \langle Px, y\rangle = 0, \text{ for all } y \in H\}.$$

But if $\langle Px, y\rangle = 0$ for all y, then $Px = 0$. Therefore, $M^\perp = K$. This tells us that the kernel is closed, since it is the orthogonal space to a subspace.

Now we know we can write $H = K + K^\perp$. If we can show that in fact $H = K + M$, then we will have $M = K^\perp$, and thus will be closed. Let $Y : H \longrightarrow M$ and $Z : H \longrightarrow M^\perp$ be the orthogonal projections. Let $x \in H$. Then $x = Yx + Zx$. Now apply P to both sides. $Px = P(Yx) + P(Zx) = P(Yx) + 0$. But $Yx = Px_0$ for some $x_0 \in H$. Thus, $P(x - Px_0) = 0$. Therefore, $x - Px_0 \in K$. So, we can write $x = (x - Px_0) + Px_0$. Thus we have decomposed $H = K + M$. ∎

Example 12. Let A be a closed subspace of $C([0,1])$ which is also closed in $L^2([0,1])$. Prove that the orthogonal projection P which projects $L^2([0,1])$ onto A is continuous as a linear transformation from $L^2([0,1])$ into $C([0,1])$. Note that $C([0,1])$ has the topology determined by the sup norm $\|\cdot\|_\infty$, while $L^2([0,1])$ has the topology determined by the norm $\|\cdot\|_2$.

Proof. Let $A_2 = (A, \|\cdot\|_2)$ and $A_\infty = (A, \|\cdot\|_\infty)$. Then we want to verify that

$$\widehat{P} : L^2([0,1]) \xrightarrow{P} A_2 \xrightarrow{\mathrm{id}} A_\infty \xrightarrow{\mathrm{incl}} C([0,1])$$

is continuous, where id is the identity map, and incl is the inclusion map. Since projection and inclusion under the same norm are always continuous, we simply need to verify that the identity function, $\mathrm{id} : A_2 \longrightarrow A_\infty$, is continuous.

We notice that $\|f\|_\infty < M$ implies that $\|f\|_2 < M$. Therefore, the inverse image of open balls are open balls. Thus, the identity map is continuous, and $\widehat{P}$ is continuous. ∎

Theorem (Riesz Representation Theorem). If λ is a continuous linear functional on a Hilbert space H, then there exists a unique $y \in H$ so that

$$\lambda x = \langle x, y\rangle, \quad \forall x \in H.$$

Theorem. Let $T \subseteq \mathbb{C}$ be the unit circle, $f \in C(T)$, and $\varepsilon > 0$. Then there exists a trigonometric polynomial,

$$P(t) = a_0 + \sum_{n=1}^{N} (a_n \cos(nt) + b_n \sin(nt))$$

such that $|f(t) - P(t)| < \varepsilon$ for all $t \in \mathbb{R}$. (i.e., the trigonometric polynomials are dense in $C(T)$).

Fourier Series

Definition (Fourier coefficients). Let $f \in L^1(T)$. We define the Fourier coefficients of f by:

$$\widehat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int}\, dt, \quad \forall n \in \mathbb{Z}.$$

Definition. $\ell^2(\mathbb{Z})$ is the space of sequences in $\mathbb{Z}$ whose sums of squares is finite. That is, $(a_n) \in \ell^2(\mathbb{Z})$ if $a_n \in \mathbb{Z}$ for all n and $\sum_{n=1}^{\infty} a_n^2 < \infty$.

Thus, we get a correspondence $L^2(T) \leftrightarrow \ell^2(\mathbb{Z})$:

Theorem (Riesz-Fischer Theorem). If $\{c_n\}$ is a sequence of complex numbers such that

$$\sum_{n=-\infty}^{\infty} |c_n|^2 < \infty,$$

then there exists $f \in L^2(T)$ such that $c_n = \widehat{f}(n)$.

Theorem (Parseval Theorem).

$$\sum_{n=-\infty}^{\infty} \widehat{f}(n)\overline{\widehat{g}(n)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\overline{g(t)}\, dt \quad \forall f, g \in L^2(T),$$

and the sum converges absolutely. Furthermore, if

$$s_N = \sum_{-N}^{N} \widehat{f}(n) e^{int}, \quad N \in \mathbb{N},$$

then $\lim_{N\to\infty} \|f - s_N\|_2 = 0$.

Chapter 5

Examples of Banach Space Techniques

Joke. Q: What's yellow, normed, linear, and complete?

Ans: A Bananach space.

Definition (Banach Space). A Banach space is a normed, linear, complete metric space. If X and Y are normed linear spaces, and $T : X \longrightarrow Y$ is linear, then we can define the norm of T, $\|T\|$ by

$$\|T\| = \sup\left\{\|Tx\|_Y : x \in X,\, \|x\|_X \le 1\right\}.$$

If $\|T\| < \infty$ we say T is bounded

Theorem. If X and Y are normed linear spaces, and $T : X \longrightarrow Y$ is linear, then the following are equivalent:

1. T is bounded
2. T is continuous
3. T is continuous at a point $x_0 \in X$.

The Big Four

Theorem (Banach-Steinhaus Theorem). Let X be a Banach space, Y a normed linear space, $\{\lambda_\alpha : X \longrightarrow Y\}$ a collection of bounded linear transformations where α ranges over some index set $\mathscr{A}$. Then either

1. There is a bound $M < \infty$ such that $\|\lambda_\alpha\| < M$ for all $\alpha \in \mathscr{A}$, or
2. There exists a dense G_δ subset of X such that

$$\sup_{\alpha \in \mathscr{A}} \|\lambda_\alpha x\| = \infty, \quad \forall x \in G_\delta.$$

Geometric Interpretation

In other words, either

1. There is a ball $B \subseteq Y$ centered at 0 with radius M so that for every $\alpha \in \mathscr{A}$, λ_α maps the unit ball of X into B, OR
2. There is an x (in fact a dense set of them) such that no ball $B \subseteq Y$ contains $\lambda_\alpha x$ for all $\alpha \in \mathscr{A}$.

Here is a weak form of the Banach-Steinhaus theorem:

Theorem. Let X be a Banach space Y a normed linear space, and let $(\lambda_\alpha) : X \longrightarrow Y$ be a set of bounded linear transformations from X to Y. Suppose there exists an $M > 0$ such that for every $\alpha \in \mathscr{A}$, $|\lambda_\alpha| \leq M$. Then for every $x \in X$, the family $\{\|\lambda_\alpha(x)\|\}$ is bounded in Y. That is,

$$\sup_\alpha \|\lambda_\alpha(x)\| < \infty.$$

The Banach-Steinhaus theorem actually says that the implication indeed goes both ways.

Theorem (Open Mapping Theorem). If X and Y are Banach spaces, and $f : X \longrightarrow Y$ is bounded, linear, and onto, then f is an open map. (The image of every open ball in X with center at x_0 contains an open ball in Y with center $f(x_0)$).

Example 13. A bounded linear transformation $T : X \longrightarrow Y$ between Banach spaces is said to be a compact linear transformation if the image under T of the unit ball in X has compact closure in Y. Prove that if a compact linear transformation is also surjective, then Y must be finite dimensional (we may assume that a locally compact Banach space must be finite dimensional).

Proof. Let $T : X \longrightarrow Y$ be a compact linear transformation between Banach spaces which is also surjective. Then T is bounded by the definition.[1] Therefore, by the open mapping theorem, T is an open map. So consider the following.

Let $y \in Y$. Then there exists $x \in X$ such that $Tx = y$. Let V_x be the ball of radius 1 about x in X. Then, $T(V_x)$ is a neighborhood of y, since T is an open map. Let $W = T(V_x)$, and $\overline{W}$ is compact, since T is a compact map.

Therefore, every point in Y has a neighborhood with compact closure. Hence, Y is locally compact, and must be finite dimensional. □

Example 14. Let $\alpha(x, y)$ be a continuous function on $[0, 1] \times [0, 1]$. Show that

$$A(f)(x) = \int_0^1 \alpha(x, y) f(y)\, dy$$

defines a bounded operator from $L^2([0, 1])$ to $C([0, 1])$, considered as a Banach space with respect to $\|f\| = \sup_x |f(x)|$.

[1]Note: we can deduce boundedness since the image of the unit ball in X is bounded in Y.

Proof. We first note that the integral defining Af does not depend on x. Thus, we can pull limits in x inside. Furthermore, since α is continuous in x, we can pass limits through α as well:

$$\begin{aligned}\lim_{x\to p} Af(x) &= \lim_{x\to p}\int_0^1 \alpha(x,y)f(y)\,dy\\ &= \int_0^1 \lim_{x\to p}\alpha(x,y)f(y)\,dy\\ &= \int_0^1 \alpha(p,y)f(y)\,dy\\ &= Af(p).\end{aligned}$$

Hence, $Af \in C([0,1])$. Now to show that it is bounded. First, we note that $f \in L^2([0,1])$ implies $f \in L^1([0,1])$ by the Schwarz inequality and the fact that $m([0,1]) < \infty$. Furthermore, $\|f\|_1 \le \|f\|_2$. Let $M_\alpha = \max\limits_{[0,1]\times[0,1]} |\alpha|$. Then we have

$$\begin{aligned}|Af(x)| &= \left|\int_0^1 \alpha(x,y)f(y)\,dy\right|\\ &\le \int_0^1 |\alpha(x,y)|\,|f(y)|\;dy\\ &\le M_\alpha\|f\|_1 < \infty.\end{aligned}$$

Hence, A is bounded. □

Definition. Let $f : X \longrightarrow Y$ be any function. Define the graph of f as

$$\text{graph}(f) = \{(x, f(x)) \in X \times Y\}.$$

If X and Y are normed, then we say that graph(f) is closed if for every sequence $\{x_n\}$ in X for which $x = \lim x_n$, and $y = \lim f(x_n)$ exist, it is true that $y = f(x)$.

Theorem (Closed Graph Theorem). If X and Y are Banach spaces, $f : X \longrightarrow Y$ linear, then f is bounded (therefore continuous) if and only if graph(f) is closed in $X \times Y$.

And now the theorem which does not depend on completeness!

Theorem (Hahn-Banach Theorem). If $Y \subseteq X$ are normed linear spaces, then for all $f \in Y^*$ there is a (nonunique) extension $F \in X^*$ so that $F\big|_Y = f$ and $\|f\| = \|F\|$.

Some Consequences of the Hahn-Banach Theorem

Theorem. Let M be a linear subspace of a normed linear space X and let $x_0 \in X$. Then x_0 is in the closure $\overline{M}$ of M if and only if there is **no** bounded linear functional f on X with $f\big|_M \equiv 0$ and $f(x_0) \neq 0$.

Theorem. If X is a normed linear space, with $x_0 \in X$, $x_0 \neq 0$ then there is a bounded linear functional f on X with norm 1 and $f(x) = \|x_0\|$.

Remark. For $x \in X$, $\|x\| = \sup\{|f(x)| : f \in X^*, \|f\| = 1\}$. Therefore, the "evaluation" map $\widehat{x} \in (X^*)^*$, given by $\widehat{x} : f \mapsto f(x)$, is a bounded linear functional on X^* of norm $\|x\|$. Hence the following Theorem:

Theorem. X embeds isometrically into X^{**}.

Example 15. Let λ be a bounded linear functional on a linear subspace V of a Hilbert space H. Show that there exists a **unique** bounded linear functional Λ on H such that

1. $\|\Lambda\| = \|\lambda\|$, and
2. $\Lambda(x) = \lambda(x)$ for all $x \in V$.

Proof. Let $\overline{V}$ be the closure of V in H. Then λ extends to $\overline{V}$ uniquely, so we may assume V is closed. By Riesz representation, λ is given uniquely by inner product. That is, there exists $w \in V$ such that $\lambda(v) = \langle v, w\rangle$ for all $v \in V$. Now, use Hahn-Banach to extend λ to Λ on H. Again, by Riesz representation, there is a $\widetilde{w} \in H$ such that $\Lambda(x) = \langle x, \widetilde{w}\rangle$ for all $x \in H$. Furthermore, $\|\Lambda\| = \|\lambda\|$, which implies $\|\widetilde{w}\| = \|w\|$. For all $x \in V$ we have $\Lambda(x) = \lambda(x)$. This implies that $\widetilde{w} - w$ is orthogonal to w. That is, $\widetilde{w} = w + z$ where $z \perp w$. But since $\|\widetilde{w}\| = \|w\|$ we may conclude that $z = 0$. $\square$

Example 16. Let X be a normed linear space.

1. Suppose (x_n) is a sequence in X such that $x_n \to x$. Show that $\lambda(x_n) \to \lambda(x)$ for all bounded linear functionals λ.

   Proof. This is simply continuity. $\square$

2. Now suppose (y_n) is a sequence in X and $\lambda(y_n) \to 0$ for all bounded linear functionals λ. Show that (y_n) is a bounded sequence.

   Proof. Let $Y = X^*$. Then Y is Banach (even if X is not!). Now we will think of each y_n as the "evaluation" functional $\widetilde{y}_n$ on Y given by $\widetilde{y}_n(\lambda) = \lambda(y_n)$. Then by Hahn-Banach, $\|\widetilde{y}_n\|_* = \|y_n\|$. Then, by assumption, $\widetilde{y}_n(\lambda) \to 0$ for every $\lambda \in Y$. But convergent sequences are bounded. Therefore, by Banach-Steinhaus, there exists $M > 0$ such that $\|\widetilde{y}_n\|_* \leq M$. Hence $\|y_n\| \leq M$. $\square$

3. Does it follow that $y_n \to 0$?

Chapter 6

Complex Measures

Definition. Let μ be a positive measure on $(X, \mathfrak{M})$, and λ and κ be complex measures. Then we say that λ is absolutely continuous with respect to μ, and write

$$\lambda \ll \mu,$$

if $\lambda(E) = 0$ whenever $\mu(E) = 0$.

We say that λ is concentrated on A if $\lambda(B) = 0$ whenever $B \cap A = \varnothing$. We say that λ and κ are mutually singular if there exist sets $L, K \subseteq X$, which cover X such that λ is concentrated on L and κ is concentrated on K. In this case, we write

$$\lambda \perp \kappa.$$

Theorem (Radon-Nikodym Theorem). Let μ be a positive σ-finite measure on X, λ a complex measure on X.

1. There are unique complex measures λ_{ab} and λ_{sing} on X such that

$$\lambda = \lambda_{\text{ab}} + \lambda_{\text{sing}}, \quad \lambda_{\text{ab}} \ll \mu, \quad \lambda_{\text{sing}} \perp \mu.$$

2. There is a unique $h \in L^1(\mu)$ such that

$$\lambda_{\text{ab}}(E) = \int_E h \, d\mu, \quad \forall E \in \mathfrak{M}, \text{(the } \sigma\text{-algebra on } X).$$

$$d\lambda_{\text{ab}} = h \, d\mu.$$

h is called the "Radon-Nikodym derivative" of λ with respect to μ.

Some Consequences of the Radon-Nikodym Theorem

Theorem. Let μ be a complex measure on X, then there exists a measurable function h on X such that $|h(x)| = 1$ for all $x \in X$ and

$$d\mu = h\, d|\mu|.$$

Theorem. Let μ be a positive measure on X, $g \in L^1(\mu)$. Then,

$$\lambda(E) = \int_E g\, d\mu \quad \Longrightarrow \quad |\lambda|(E) = \int_E |g|\, d\mu.$$

Theorem (Hahn Decomposition Theorem). Let μ be a real measure on X, then there exist measurable disjoint sets A and B which cover X and

$$\mu^+(E) = \mu(A \cap E), \quad \mu^-(E) = -\mu(B \cap E),$$

where

$$\mu^+ = \frac{1}{2}(|\mu| + \mu), \quad \mu^- = \frac{1}{2}(|\mu| - \mu).$$

Bounded Linear Functionals on $L^p(\mu)$

Let μ be a positive measure, p and q conjugate exponents, $1 \leq p \leq \infty$. For all $g \in L^q(\mu)$, define $\overline{g} : L^p(\mu) \longrightarrow \mathbb{R}$ by

$$\overline{g}(f) = \int_X fg\, d\mu.$$

Then Hölder's Inequality implies $\overline{g} \in L^p(\mu)^*$ is bounded with norm at most $\|g\|_q$. In fact:

Theorem. If $p < \infty$ and $\phi \in L^p(\mu)^*$ is bounded, then there exists a unique $g \in L^q(\mu)$ with $\phi = \overline{g} = \int_X \cdot\, g\, d\mu$, and $\|\phi\| = \|g\|_q$.

In other words, $L^p(\mu)^*$ is isometrically equivalent to $L^q(\mu)$.

Theorem (Riesz Representation Theorem). If X is a locally compact Hausdorff space then every linear functional ϕ on $C_0(X)$ is represented by a unique regular complex Borel measure μ via

$$\phi(f) = \int_X f\, d\mu, \quad \forall f \in C_0(X).$$

Moreover, the norm of ϕ is the total variation of μ:

$$\|\phi\| = |\mu|(X).$$

In words, there is an isometry $C_0(X)^* \cong \mathscr{B}(X)$, the space of regular complex Borel measures on X.

Example 17. Let $X = [0,1]$, and $C(X)$ denote the space of continuous complex valued functions on X, equipped with the sup norm:

$$\|f\|_\infty = \sup_{x \in [0,1]} |f(x)|.$$

Let f_n be a sequence in $C(X)$. Prove that the following statements are equivalent.

1. For each bounded linear functional λ on $C(X)$, $\lambda(f_n) \to 0$;
2. $f_n(x) \to 0$ for all $x \in X$, and $\sup_n \|f\|_\infty$ is finite.

Proof. Suppose that $\lambda(f_n) \to 0$ for every bounded linear functional λ on $C(X)$. Then, for each $x \in X$, the evaluation map $\widetilde{x} : f \mapsto f(x)$ is bounded and linear. Thus, $f_n(x) = \widetilde{x}(f_n) \to 0$. Since f_n are all continuous on a compact set X, it is clear that $\sup_n \|f_n\|_\infty$ must be finite if $f_n \to 0$. Suppose not. We will find a point x_0 where $f_n(x_0) \not\to 0$. For $m \in \mathbb{N}$, there is some $N_m > 0$ such that for each $n > N_m$ there is a point $y_m(n) \in X$ such that $|f_n(y_m(n))| > m$. Let $n_m = \min\{n \in \mathbb{N} : n > N_m\}$, and let $x_m = y_m(n_m)$. Then x_m forms a sequence of points in X. Since X is compact, there is some $x_0 \in X$ such that $x_m \to x_0$. Then, since all of the f_n are continuous, the sequence $f_n(x_0)$ diverges. This is a contradiction.

Conversely, Suppose $f_n \to 0$ and $M = \sup_n \|f\|_\infty$ is finite. Let λ be a bounded linear functional on $C(X)$. Then by the Riesz representation theorem, λ is represented by a regular Borel measure μ in that

$$\lambda(f) = \int_0^1 f(x)\, d\mu(x).$$

Thus, we apply Lebesgue dominated convergence theorem to the f_n, which are dominated by the constant function M. We conclude that $\lambda(f_n) \to 0$. ☐

The following problem will pull together many of the important ideas.

Example 18. Let D be the unit disk in $\mathbb{C}$, $\overline{D}$ its closure, and T its boundary, the unit circle. Prove that for every positive measure μ of total mass one on $\overline{D}$ there is a positive measure ν of total mass one on T for which

$$\int_D f\, d\mu = \int_T f\, d\nu$$

for every f which is continuous on $\overline{D}$ and holomorphic on D.

Proof. Let $X = C(\overline{D}) \cap \mathscr{H}(D)$. Then X is a Banach space under the sup-norm

$$\|f\| = \sup_{z \in \overline{D}} |f(z)| = \sup_{z \in T} |f(z)|$$

(the second equality follows from the Maximum Modulus Principle). Let $r : C(\overline{D}) \longrightarrow C(T)$ be the restriction map $f \mapsto f\big|_T$. r is not injective, but if we restrict to X, it is. Furthermore, by the Maximum Modulus Principle, $\|rf\|_{C(T)} = \|f\|_X$. Therefore, X and $r(X)$ are isometrically equivalent.

Now let μ be a positive measure on $\overline{D}$ of total mass one. Let Λ denote integration with respect to μ. That is,

$$\Lambda(f) = \int_D f\, d\mu$$

for all $f \in C(\overline{D})$. Then $\Lambda \in C(\overline{D})^*$ and $\|\Lambda\| = |\mu|(D) = \mu(D) = 1$. Now restrict Λ to X. So we think of $\Lambda\big|_X \in X^*$. It is still bounded. In fact, $\|\Lambda\big|_X\| = 1$ as well!

So we have this isometry $r : X \longrightarrow r(X) \subseteq C(T)$, and we have a bounded linear functional $\Lambda\big|_X$ on X. Define $\varphi \in r(X)^*$ via

$$\varphi(f) = \Lambda(r^{-1}f)$$

for all $f \in r(X)$. Then $\|\varphi\| = 1$.

By Hahn-Banach, we can extend φ to $\Phi \in C(T)^*$ so that $\|\Phi\| = \|\varphi\|$. Then Φ is a bounded linear functional on the space of continuous functions over a compact Hausdorff space. Thus, by Riesz Representation, there exists a regular complex Borel measure ν of total variation $\|\Phi\| = 1$, such that

$$\Phi(f) = \int_T f\, d\nu.$$

But

$$|\nu|(T) = 1 = \Phi(1) = \int_T d\nu = \nu(T).$$

Therefore, in fact, ν is positive. □

Example 19. Let $I = [0,1]$, and $T = \{0,1\}$. Let X be the space of real linear functions on I. Let dx be the Lebesgue measure on I. Then this problem is identical to the previous problem (all the arguments hold, with only slight modification). Therefore, there is some positive measure ν on T of total mass one so that

$$\int_I f\, dx = \int_T f\, d\nu$$

for all $f \in X$. Using ordinary calculus, we may see that $\int_0^1 ax + b\, dx = \frac{a}{2} + b$. Let $\nu(0) = \nu(1) = \frac{1}{2}$. Then integration over T is simply the average of the values at the end points. In other words,

$$\int_{\{0,1\}} ax + b\, d\nu = (a+b)\nu(1) + b\nu(0) = \frac{a}{2} + b,$$

which is what we hoped for.

Chapter 7

Differentiation

Chapter 8

Integration on Product Spaces

Theorem (Fubini's Theorem).

$$\int_X d\mu(x) \int_Y f(x,y)\, d\lambda(y) = \int_Y d\lambda(y) \int_X f(x,y)\, d\mu(x),$$

when X and Y are σ-finite measure spaces with measures μ and λ, respectfully.

Example 20. Let $(X, \mathfrak{M}, \mu)$ be a positive measure space, and let $f \geq 0$ be a measurable function on X. Define

$$g(t) = \mu\left(\{x \in X \,:\, f(x) \geq t\}\right).$$

Show that

$$\int_X f(x)\, d\mu(x) = \int_0^\infty g(t)\, dt.$$

Proof. Define $E(t) = \{x \in X \,:\, f(x) \geq t\}$. Then $g(t) = \mu(E(t))$, and for a fixed t,

$$\chi_{E(t)}(x) = \begin{cases} 1, & \text{if } t \leq f(x), \\ 0, & \text{if } t > f(x). \end{cases}$$

Therefore,

$$\int_0^\infty g(t)\, dt = \int_0^\infty \int_X \chi_{E(t)}(x)\, d\mu(x) dt.$$

Applying Fubini's theorem to the right hand side yields

$$\int_0^\infty g(t)\, dt = \int_X d\mu(x) \int_0^\infty \chi_{E(t)}(x)\, dt.$$

However, for the inner integral, we need only integrate up to $t = f(x)$, since beyond that point $\chi_{E(t)}(x)$ is identically zero. Furthermore, on $[0, f(x)]$, the integrand is identically 1. Then we make use of the Fundamental Theorem of Calculus:

$$\begin{aligned}
\int_0^\infty g(t)\,dt &= \int_X d\mu(x) \int_0^\infty \chi_{E(t)}(x)\,dt \\
&= \int_X d\mu(x) \int_0^{f(x)} dt \\
&= \int_X f(x)\,d\mu(x).
\end{aligned}$$

The result holds. $\square$

Definition (Convolution). Let $f, g \in L^1(\mathbb{R})$. Define their convolution product, $f * g$ by

$$f * g(x) = \int_{-\infty}^{\infty} f(x-\xi) g(\xi)\,d\xi, \quad \forall x \in \mathbb{R}.$$

Then with $*$, $L^1(\mathbb{R})$ is a Banach $\mathbb{R}$-algebra ($\|f * g\|_1 \leq \|f\|_1 \, \|g\|_1$).

Chapter 9

Fourier Transforms

Let $dm(x) = \frac{1}{\sqrt{2\pi}} dx$. This simplifies some of the formulations.

Definition (Fourier Transform). For every $f \in L^1(\mathbb{R})$ define its Fourier transform, $\widehat{f}$ by

$$\widehat{f}(t) = \int_{-\infty}^{\infty} f(x) e^{-ixt}\, dm(x), \quad \forall t \in \mathbb{R}.$$

Theorem (Properties of the Fourier Transform). Let $f \in L^1(\mathbb{R})$, and $a, b \in \mathbb{R}$. Then

1. If $g(x) = f(x)e^{iax}$ then $\widehat{g}(t) = \widehat{f}(t-a)$,
2. If $g(x) = f(x-a)$ then $\widehat{g}(t) = \widehat{f}(t)e^{-iat}$,
3. If $g \in L^1(\mathbb{R})$, $h = f * g$, then $\widehat{h} = \widehat{f}\,\widehat{g}$,
4. If $g(x) = \overline{f(-x)}$, then $\widehat{g} = \overline{\widehat{f}}$,
5. If $g(x) = f(x/b)$ with $b > 0$, then $\widehat{g}(t) = b\widehat{f}(bt)$,
6. If $g(x) = -ixf(x)$ and $g \in L^1(\mathbb{R})$, then $\widehat{f}$ is differentiable with $\left(\widehat{f}\right)' = \widehat{g}$.
7. If $f' \in L^1(\mathbb{R})$, then $\widehat{(f')}(t) = it\widehat{f}(t)$,
8. $\widehat{f} \in C_0(\mathbb{R})$ and
$$\|\widehat{f}\|_\infty \le \|f\|_1.$$

Theorem (Inversion Theorem). If $f \in L^1(\mathbb{R})$, $\widehat{f} \in L^1(\mathbb{R})$, and if we define

$$g(x) = \int_{-\infty}^{\infty} \widehat{f}(t) e^{ixt}\, dm(t), \quad \forall x \in \mathbb{R},$$

then $g \in C_0$ and $f(x) = g(x)$ a.e.

Corollary (Uniqueness Theorem). If $f \in L^1(\mathbb{R})$, $\widehat{f} \equiv 0$, then $f(x) = 0$ a.e.

Proof. $\widehat{f} \equiv 0 \implies \widehat{f} \in L^1(\mathbb{R})$. Apply the Inversion Theorem. $\square$

We can summarize all of these results with the following.

Theorem. $\widehat{\ }\colon L^1(\mathbb{R}) \longrightarrow C_0(\mathbb{R})$ is a monomorphism of Banach Algebras, where $L^1(\mathbb{R})$ has the L^1-norm and multiplication given by convolution, while $C_0(\mathbb{R})$ has the sup-norm and multiplication given by ordinary multiplication.

Example 21. Let f be a continuous positive valued function on $\mathbb{R}$, and let g be the characteristic function of a finite nonempty open interval (a, b) in $\mathbb{R}$. Show that the function $h = fg$ is in $L^1(\mathbb{R})$ but its Fourier transform $\widehat{h}$ is not.

Proof. Let $M = \max\limits_{x\in[a,b]} f(x)$. Note that M exists and is finite since $f \in C(\mathbb{R})$. Then

$$\int_{\mathbb{R}} |h| = \int_a^b f(x)\,dx \leq M(b-a) < \infty.$$

Thus, $h \in L^1(\mathbb{R})$.

Conversely, choose $(a, b) = (0, 1)$, and let $g = \chi_{(0,1)}$ and $f(x) = \sqrt{2\pi}$, the constant function. Then,

$$\widehat{h}(t) = \int_0^1 e^{-ixt}\,dx = -it\left[e^{-it} - 1\right].$$

Therefore,

$$\left|\widehat{h}(t)\right| = |t|\left|1 - e^{-it}\right| = |t||1 + \sin t - \cos t|.$$

Clearly, integrating $\left|\widehat{h}\right|$ over $\mathbb{R}$ yields ∞. Thus, $\widehat{h} \notin L^1(\mathbb{R})$. $\square$

Example 22. Suppose that $f \in L^1(\mathbb{R})$ and $\widehat{f} \in L^1(\mathbb{R})$. Show that $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

Proof. Since f and $\widehat{f}$ are both in $L^1(\mathbb{R})$, we may apply the Inversion theorem, to obtain a function $g \in C_0(\mathbb{R})$ which agrees with f almost everywhere. Thus, without loss of generality, it suffices to show that $g \in L^2(\mathbb{R})$.

Since $g \in C_0(\mathbb{R})$, there is a finite interval (possibly empty) $[a, b]$ outside of which $|g(x)| < 1$. Furthermore, $|g|$ attains its maximum value M on $[a, b]$, so that

$$\begin{aligned}
\int_{\mathbb{R}} |g(x)|^2\,dx &= \int_a^b |g(x)|^2\,dx + \int_{\mathbb{R}\setminus[a,b]} |g(x)|^2\,dx \\
&\leq \int_a^b M^2 + \int_{\mathbb{R}\setminus[a,b]} |g(x)|\,dx \\
&\leq M^2(b-a) + \|f\|_1 < \infty.
\end{aligned}$$

Note that by the Parseval Theorem, below, we can say that

$$\widehat{f} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$$

as well! $\square$

Though we cannot extend, directly, the definition of Fourier transform to $L^2(\mathbb{R})$, we have the Parseval transform:

Theorem (Existence of Parseval Transform). To each $f \in L^2(\mathbb{R})$, one can associate a function $\widehat{f} \in L^2(\mathbb{R})$ such that:

1. If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then $\widehat{f}$ is its Fourier transform, as above,
2. $\|\widehat{f}\|_2 \leq \|f\|_2$,
3. The mapping $f \mapsto \widehat{f}$ is a Hilbert space isomorphism of $L^2(\mathbb{R})$ onto itself. Therefore, there is an inversion!
4. If we define
$$\varphi_A(t) = \int_{-A}^{A} f(x)e^{-ixt}\, dm(x), \quad \forall t \in \mathbb{R},$$
and
$$\psi_A(x) = \int_{-A}^{A} \widehat{f}(t)e^{ixt}\, dm(t), \quad \forall x \in \mathbb{R},$$
then $\|\varphi_A - \widehat{f}\|_2 \to 0$ and $\|\psi_A - f\|_2 \to 0$ as $A \to \infty$,
5. If $\widehat{f} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, and we define
$$g(x) = \int_{-\infty}^{\infty} \widehat{f}(t)e^{ixt}\, dm(t), \quad \forall x \in \mathbb{R},$$
then $g = f$ a.e.

Chapter 10

Elementary Properties of Holomorphic Functions

Theorem. Let γ be a closed path, and let Ω be the complement of γ (Here we are abusing notation, by identifying γ with its image). Define the index of γ about the point $z \in \Omega$ by

$$\mathrm{Ind}_\gamma(z) = \frac{1}{2\pi i} \int_\gamma \frac{d\xi}{\xi - z}, \quad \forall z \in \Omega.$$

Then, Ind_γ is an integer-valued function, constant on connected components of Ω, and 0 on the unbounded component of Ω.

$\mathrm{Ind}_\gamma(z)$ is the same as the winding number of γ about z.

Theorem. If γ is a positively oriented simple closed curve, and Ω is the complement of γ, then $\mathrm{Ind}_\gamma \equiv 1$ on the bounded component of Ω and (by the above theorem) $\mathrm{Ind}_\gamma \equiv 0$ on the unbounded component. Furthermore, $\mathrm{Ind}_{-\gamma} = -\mathrm{Ind}_\gamma$.

Theorem. If $f \in \mathscr{H}(\Omega)$ and $f' \in C(\Omega)$ then

$$\int_\gamma f'(z)\, dz = 0$$

for every closed path γ in Ω.

Theorem (Cauchy's Theorem for a Triangle). Suppose Δ is a closed triangle in a plane open set Ω, $p \in \Omega$, $f \in C(\Omega)$, and $f \in \mathscr{H}(\Omega - \{p\})$. Then,

$$\int_{\partial\Delta} f(z)\, dz = 0.$$

Theorem (Cauchy's Theorem for a Convex Set). Suppose Ω is a convex open set, $p \in \Omega$, $f \in C(\Omega)$, and $f \in \mathscr{H}(\Omega \setminus \{p\})$. Then there exists $F \in \mathscr{H}(\Omega)$ such that $f = F'$, hence,

$$\int_\gamma f(z)\,dz = 0,$$

for every closed path γ in Ω.

And we have a partial converse to Cauchy's Theorem on Triangles:

Theorem (Morera's Theorem). Suppose Ω is open, $f \in C(\Omega)$, and

$$\int_{\partial\Delta} f(z)\,dz = 0$$

for every closed triangle $\Delta \subseteq \Omega$. Then $f \in \mathscr{H}(\Omega)$.

Example 23. Suppose f is continuous on $U = \{z \in \mathbb{C} : |z| < 1\}$ and holomorphic on the complement in U of the interval $I = (-1, 1)$. Prove that f is actually holomorphic on U.

Proof. By Morera's theorem, it suffices to show that for every closed triangle Δ in U,

$$\int_{\partial\Delta} f(z)\,dz = 0.$$

By Cauchy's theorem, we need not check triangles which do not intersect I.

First we will consider triangles such as A in Figure 10.1, which meet I in exactly one point. Then by continuity of f, the point p of intersection is a removable singularity. Thus the integral around this triangle will be zero (again, by Cauchy's theorem).

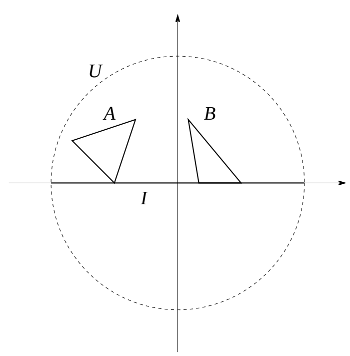

Figure 10.1: Triangles which meet I at a point or along an edge.

Now consider a triangle such as B in Figure 10.1, which meets I along one of its edges. Let Δ_n be a sequence of triangles all contained in $U \setminus I$ whose limit is B. Then by continuity of f,

$$\int_{\partial B} f(z)\, dz = \lim_{n\to\infty} \int_{\partial \Delta_n} f(z)\, dz = 0.$$

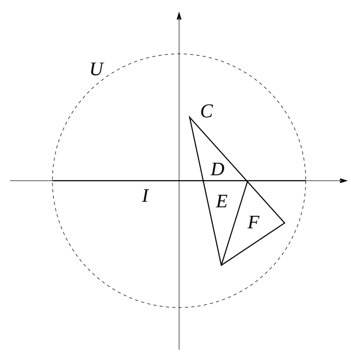

Figure 10.2: Triangles which meet I through their interior.

Lastly, we can consider triangles such as C in Figure 10.2, to be a sum of triangles like A and B as indicated, $C = D + E + F$. □

Theorem. Analytic is equivalent to Holomorphic

Corollary. If $f \in \mathscr{H}(\Omega)$ then $f' \in \mathscr{H}(\Omega)$.

Power Series Representation

Theorem. Suppose Ω is a region, $f \in \mathscr{H}(\Omega)$ and we define the "zero set of f"

$$Z(f) = \left\{ x \in \Omega \;\middle|\; f(z) = 0 \right\}.$$

Then either

1. $Z(f) = \Omega \qquad (f \equiv 0)$, or
2. $Z(f)$ has no limit point in Ω.

In the second case, there corresponds to each $a \in Z(f)$ a unique positive integer $m = m(a)$ such that

$$f(z) = (z - a)^m g(z)$$

where $g \in \mathscr{H}(\Omega)$ and $g(a) \neq 0$; furthermore, $Z(f)$ is at most countable.

Corollary. If $f, g \in \mathscr{H}(\Omega)$ and $f(z) = g(z)$ for all $z \in S \subseteq \Omega$ where S has a limit point in Ω, then $f \equiv g$ on all Ω.

Proof. Apply the above theorem to $h = f - g$. □

Definition (Singularities). If $p \in \Omega$, $f \in \mathscr{H}(\Omega \smallsetminus \{p\})$, then f has an isolated singularity at p. If f can be so defined at p that the extended function is holomorphic on Ω, then p is called a removable singularity.

Theorem. Let $p \in \Omega$, $f \in \mathscr{H}(\Omega \smallsetminus \{p\})$. Then one of the following must occur:

1. p is a removable singularity.

2. p is a pole of order m:
   There exists $m \in \mathbb{N}$, $\{c_1, \ldots, c_m\} \subseteq \mathbb{C}$, $c_m \neq 0$, such that

$$f(z) - \sum_{k=1}^{m} \frac{c_k}{(z-p)^k}$$

   has a removable singularity at p.

3. p is an essential singularity: For every $w \in \mathbb{C}$, there exists a sequence $(z_n) \subseteq \mathbb{C}$ such that $z_n \to p$ and $f(z_n) \to w$ as $n \to \infty$.

Theorem (Liouville Theorem). Every bounded entire function is constant

Theorem (Maximum Modulus Principle). Suppose Ω is a region, $f \in \mathscr{H}(\Omega)$, and $\overline{D(a,r)} \subseteq \Omega$. Then

$$|f(a)| \leq \max_{\theta} \left| f\left(a + re^{i\theta}\right)\right|.$$

Equality occurs if and only if f is constant.

Theorem (Minimum Modulus Principle). Moreover, if f has no zeros in $\overline{D(a,r)}$, then

$$|f(a)| \geq \min_{\theta} \left| f\left(a + re^{i\theta}\right)\right|.$$

Theorem (Cauchy Estimates). If $f \in \mathscr{H}(D(a,R))$ and $|f(z)| \leq M$ for all $z \in D(a,R)$, then

$$\left|f^{(n)}(a)\right| \leq \frac{n!M}{R^n}, \quad \forall n \in \mathbb{N}.$$

Lemma. If $f \in \mathscr{H}(\Omega)$ and g is defined on $\Omega \times \Omega$ by

$$g(z,w) = \begin{cases} \dfrac{f(z)-f(w)}{z-w}, & \text{if } z \neq w \\ f'(z), & \text{if } z = w, \end{cases}$$

then g is continuous on $\Omega \times \Omega$.

Proof. Obviously we need only check this on the diagonal. Let $a \in \Omega$, and let $\varepsilon > 0$. Then there exists an $r > 0$ such that $D(a,r) \subseteq \Omega$ and

$$|f'(\zeta) - f'(a)| < \varepsilon,$$

for all $\zeta \in D(a,r)$. If $z, w \in D(a,r)$, then the line segment

$$\zeta(t) = tw + (1-t)z$$

for $t \in [0,1]$ is also in $D(a,r)$. Now

$$\begin{aligned} g(z,w) - g(a,a) &= \frac{f(z)-f(w)}{z-w} - f'(a) \\ &= \int_0^1 f'(\zeta(t))\ dt - f'(a)\int_0^1 dt \\ &= \int_0^1 [f'(\zeta(t)) - f'(a)]\ dt. \end{aligned}$$

Since the integrand is less than ε in absolute value, we may conclude that

$$|g(z,w) - g(a,a)| < \varepsilon.$$

□

Theorem. Suppose $\varphi \in \mathscr{H}(\Omega)$, $z_0 \in \Omega$, and $\varphi'(z_0) \neq 0$. Then Ω contains an open neighborhood V of z_0 such that

1. φ is injective in V,
2. $W := \varphi(V)$ is open, and
3. If $\psi : W \longrightarrow V$ is defined by $\psi(\varphi(z)) = z$, then $\psi \in \mathscr{H}(W)$.

(Thus, φ locally has a holomorphic inverse).

Theorem. Suppose Ω is a region, $f \in \mathscr{H}(\Omega)$, f is not constant, $z_0 \in \Omega$ and $w_0 = f(z_0)$. Let m be the order of the zero that the function $f(z) - w_0$ has at the point z_0.

Then there is an open neighborhood $V \subseteq \Omega$ of z_0 and $\varphi \in \mathscr{H}(V)$, such that

1. $f(z) = w_0 + [\varphi(z)]^m$, for all $z \in V$, and
2. φ' has no zero in V, and φ is an invertible mapping from V onto a disk $D(0,r)$.

It follows that f is an m-to-one mapping of $V \setminus \{z_0\}$ onto $D^*(w_0, r^m)$, and that each w_0 is an interior point. Thus $f(\Omega)$ is a region. This is a strong restatement of:

Theorem (The Open Mapping Theorem). If Ω is a region and $f \in \mathscr{H}(\Omega)$, then $f(\Omega)$ is either a region or a point.

Theorem. Suppose Ω is a region, $f \in \mathscr{H}(\Omega)$, and f is injective in Ω. Then f' is nowhere zero in Ω and the inverse of f is holomorphic.

The Global Cauchy Theorem

Theorem. Suppose $f \in \mathscr{H}(\Omega)$ where Ω is an arbitrary open set in the complex plane. If Γ is a cycle in Ω which satisfies

$$\mathrm{Ind}_\Gamma(\alpha) = 0, \quad \text{for every } \alpha \text{ not in } \Omega,$$

then

$$f(z) \cdot \mathrm{Ind}_\Gamma(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\xi)}{\xi - z}\, d\xi, \quad \text{for } z \in \Omega \smallsetminus \Gamma,$$

and

$$\int_\Gamma f(z)\, dz = 0.$$

Furthermore, if Γ_1 and Γ_2 are cycles with

$$\mathrm{Ind}_{\Gamma_1}(\alpha) = \mathrm{Ind}_{\Gamma_2}(\alpha), \quad \text{for every } \alpha \text{ not in } \Omega,$$

then

$$\int_{\Gamma_1} f(z)\, dz = \int_{\Gamma_1} f(z)\, dz.$$

Calculus of Residues

Definition. A function f is said to be meromorphic in an open set Ω if there is a set $A \subseteq \Omega$ such that

1. A has no limit point in Ω,
2. $f \in \mathscr{H}(\Omega \smallsetminus A)$, and
3. f has a pole at each point in A.

Note that A is possibly empty.

Definition. If f has a pole of order m at $a \in A$, then the function

$$Q(z) = \sum_{k=1}^{m} \frac{c_k}{(z-a)^m},$$

for which $f - Q$ has a removable singularity at a, is called the principle part of f at a. The complex number c_1 is called the residue of f at a, and is denoted $\mathrm{Res}(f, a)$.

Theorem (The Residue Theorem). Let f be meromorphic in Ω, and let A be the set of poles of f in Ω. If Γ is a cycle in $\Omega \smallsetminus A$ such that

$$\mathrm{Ind}_\Gamma(\alpha) = 0, \quad \text{for every } \alpha \text{ not in } \Omega,$$

then

$$\int_\Gamma f(z)\, dz = 2\pi i \sum_{a \in A} \mathrm{Res}(f, a) \cdot \mathrm{Ind}_\Gamma(a).$$

Theorem (Rouché's Theorem). Suppose Γ is a closed path in a region Ω, such that $\mathrm{Ind}_\Gamma(\alpha) = 0$ for all α not in Ω. Suppose also that $\mathrm{Ind}_\Gamma(\alpha)$ is either 0 or 1 for all $\alpha \in \Omega \smallsetminus \Gamma$; and let $\Omega_1 = \{\alpha \in \Omega \smallsetminus \Gamma \mid \mathrm{Ind}_\Gamma(\alpha) = 1\}$.

For any $f \in \mathscr{H}(\Omega)$ let N_f denote the number of zeros of f in Ω_1, counting multiplicity. Then,

1. If $f \in \mathscr{H}(\Omega)$ and f has no zeros along Γ, then
$$N_f = \frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)}\,dz = \mathrm{Ind}_\Lambda(0),$$
where $\Lambda = f \circ \Gamma$.
2. If also $g \in \mathscr{H}(\Omega)$ and
$$|f(z) - g(z)| < |f(z)|, \quad \text{for all } z \in \Gamma,$$
then $N_g = N_f$.[1]

Example 24. Determine how many zeros, counting multiplicity, the polynomial
$$f(z) = z^5 - 6z^4 + z^3 + 2z - 1$$
has in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

Solution. Let $g(z) = -6z^4$. Then g has a zero of degree 4 at the origin. On the boundary of U, $|g| = 6$, while $|f - g| \leq 5$. Thus, by Rouché's theorem, f also has 4 zeros, counting multiplicity, on U. $\square$

Example 25. Determine the number of zeros the polynomial
$$f(z) = z^5 + 4z^2 + 2z + 1$$
has in the region $\overline{U} = \{z \in \mathbb{C} : |z| \leq 1\}$

Solution. This one is a bit trickier than the last one. We cannot simply subtract off a single term. If we try $g(z) = 4z^2$, then at $z = 1$, $|f - g| = |g|$, and Rouché requires strict inequality. The other terms are even worse (try it). So we shall try two terms. Either $g(z) = z^5 + 4z^2$ or $h(z) = 4z^2 + 2z$ will work. We will demonstrate how to use $g(z)$. Certainly, on $T = \partial U$, we have $|f - g| = |2z + 1| \leq 3$ and $3 \leq |z^5 + 4z^2|$. There are two ways of looking at this: the first inequality is only equality at $z = 1$, in which case $|g| = 5$; or, just as useful, the second inequality is only equality at $z = -1$, where $|f - g| = 1$. Either way, we may conclude that $|f - g| < |g|$, apply Rouché, to get that f has two zeros on $\overline{U}$ (since $g(z) = z^2(z^3 + 4)$ has a double root at $z = 0$ and the rest outside of $\overline{U}$). $\square$

Example 26. Show that all zeros of $f(z) = z^4 - 6z - 3$ lie inside the circle $\Gamma = \{z \in \mathbb{C} : |z| = 2\}$.

[1] Note that we could equivalently use the condition that $|f(z) - g(z)| < |g(z)|$ for all $z \in \Gamma$.

Proof. Let $g(z) = z^4$. Then on Γ, $|g| = 16$, while $|f - g| \leq 15$. Thus, by Rouché's theorem, the number of zeros of f inside of Γ is the same as for g. Namely, four. But these are all the zeros of f by the Fundamental Theorem of Algebra. $\square$

Example 27. Let $\lambda > 1$. Show that the equation $\lambda - z - e^{-z} = 0$ has one solution for $\Re(z) > 0$.

Proof. Let $f(z) = \lambda - z - e^{-z}$. If $f(z) = 0$ has any solutions for $\Re(z) > 0$ then

$$|\lambda - z| = |e^{-z}| < 1.$$

So we will look at $D = D(\lambda, 1)$. Since $\lambda > 1$, $D \subseteq \Omega = \{z : \Re(z) > 0\}$. Let $g(z) = \lambda - z$. Then g has exactly one solution on D, namely $z = \lambda$. Consider

$$|f(z) - g(z)| = |e^{-z}| < 1$$

for all $z \in \partial D$, and

$$|g(z)| = |\lambda - z| = 1$$

for all $z \in \partial D$. Thus, by Rouché's theorem, f and g have the same number of zeros in D. Thus f has one zero for $\Re(z) > 0$. $\square$

Techniques for Finding Residues

In the Table 10.1, g and h are holomorphic at p, and f has an isolated singularity there.

Function	Test	Singularity	Residue at p
$f(z)$	$\lim_{z\to p}(z-p)f(z)=0$	removable	0
$f(z)$	$\lim_{z\to p}(z-p)f(z)=L\neq 0$ (exists, and is nonzero)	simple pole	L
$\dfrac{g(z)}{h(z)}$	g and h have zeros at p of the same order	removable	0
$\dfrac{g(z)}{h(z)}$	g has zero at p of order k, h has zero of order $k+m$	pole of order m	$\lim_{z\to p}\dfrac{\phi^{(m-1)}(z)}{(m-1)!}$, where $\phi(z)=(z-p)^m\dfrac{g(z)}{h(z)}$
$\dfrac{1}{h(z)}$	h has a simple zero at p	simple pole	$\dfrac{1}{h'(p)}$
$\dfrac{g(z)}{h(z)}$	g has zero at p of finite order, all derivatives of h at p are zero.	essential	Usually tricky. We must find the Laurent expansion.

Table 10.1: Summary of Tests and Residues at Isolated Singularities.

Example 28. We will prove a variation of one of the statements in Table 10.1. Let Ω be open in $\mathbb{C}$, $b\in\Omega$, $f\in\mathscr{H}(\Omega)$, and assume that $f'(b)\neq 0$. Show that

$$\frac{2\pi i}{f'(b)}=\int_C\frac{1}{f(z)-f(b)}\,dz$$

for sufficiently small positively oriented circles centered at b.

Proof. Let $h(z)=f(z)-f(b)$. Then $h'(b)=f'(b)\neq 0$. Thus h has a simple zero at b. Let $g=\dfrac{1}{h}$. We wish to show that $\operatorname{Res}(g,b)=\dfrac{1}{h'(b)}$. Using the second statement in Table 10.1, we have

$$\operatorname{Res}(g,b)=\lim_{z\to b}\frac{z-b}{h(z)}=\frac{1}{h'(b)},$$

by l'Hôpital's rule. $\square$

Example 29. Let $f(z) = \dfrac{1}{z^2}e^z$. Then f has an isolated singularity at 0. Furthermore, f looks like $f = \frac{g}{h}$ where $g(z) = e^z$ and $h(z) = z^2$. We see that g has no zero at 0, while h has a zero of degree 2. Thus, $z = 0$ is a pole of order 2 for f. Furthermore, $\text{Res}(f, 0) = \lim_{z\to 0} g'(z) = 1$. Thus, if C is the unit circle, we conclude via the Residue Theorem, that

$$\int_C f(z)\,dz = 2\pi i.$$

Example 30. Suppose we want to find the singularities and their types of the function

$$f(z) = \frac{1}{e^z - 1} - \frac{1}{z}.$$

First we can rewrite f so that it looks like $\dfrac{g}{h}$ with g and h holomorphic:

$$f(z) = \frac{z - e^z + 1}{z\,(e^z + 1)}.$$

Thus we see that the singularities of f are at the zeros of h. Namely they occur at $p = 2n\pi i$, where $n \in \mathbb{Z}$. We will consider $p = 0$ and $p \neq 0$ as separate cases.

If $p \neq 0$ then we have $h'(p) = p \neq 0$. Furthermore $g(p) = p \neq 0$. Thus, all of the poles of this type are simple.

Now consider $p = 0$. Then $g(0) = g'(0) = 0$, $g''(0) = -1$; and $h(0) = h'(0)$, $h''(0) = 3$. Thus since g and h both have a degree 2 zero at $p = 0$ we conclude that $p = 0$ is a removable singularity. Furthermore, $\lim_{z\to 0} f(z) = \dfrac{g''(z)}{h''(z)} = -\dfrac{1}{3}$, by l'Hôpital's Rule.

Example 31. Suppose $a > 1$. Compute

$$I = \int_0^{2\pi} \frac{d\theta}{a + \sin\theta}.$$

Solution. First, we make the change of variables $z = e^{i\theta}$. Then $dz = i\theta\,d\theta$, and $\sin\theta = \dfrac{z - z^{-1}}{2i}$. Thus, the integral becomes

$$I = \int_T \frac{2z^2}{2aiz + z^2 - 1}\,dz,$$

where T is the unit circle. Let $f(z) = 2aiz + z^2 - 1$, and let $g = \frac{2}{f}$. Then g has a pole whenever f has a zero. Using the Quadratic Formula, we see that the zeros of f are

$$p_{1,2} = \frac{-2ai \pm \sqrt{-4a^2 + 4}}{2} = i\left(a \pm \sqrt{a^2 - 1}\right).$$

Only $p = p_2 = i\left(a - \sqrt{a^2-1}\right)$ lies within the unit disk, and this pole is of order 1. The residue of g at this pole is

$$\operatorname{Res}(g,p) = \frac{2}{f'(p)}.$$

Calculate $f'(z) = (z-p_1) + (z-p_2)$. Therefore $f'(p) = p_2 - p_1 = -2i\sqrt{a^2-1}$. The residue at p then becomes

$$\operatorname{Res}(g,p) = \frac{2}{f'(p)} = \frac{i}{\sqrt{a^2-1}}.$$

We can now calculate that the integral is

$$I = 2\pi i \cdot \operatorname{Res}(g,p) = \boxed{-\frac{2\pi}{\sqrt{a^2-1}}}$$

□

One trick that may be useful is a change of variables. Consider the integral

$$I = \int_\Gamma f(z)\,dz.$$

If f has only poles or removable singularities inside of Γ, we simply apply the Residue Theorem. Otherwise we may use the following:

Lemma. Let Γ be a closed simple curve in the plane. Let $\phi(z) = 1/z$ be the inversion map. Denote by $1/\Gamma$ the composition $\phi \circ \Gamma$. Then we have the following change of coordinates:

$$\int_\Gamma f(z)\,dz = \int_{1/\Gamma} \frac{1}{y^2} f\left(\frac{1}{y}\right)\,dy.$$

Proof. Let $z = \dfrac{1}{y}$. □

Example 32. Let $f(z) = e^{1/z}$. Then f has an essential singularity at 0. Let C be the unit circle, then $C = 1/C$. Therefore,

$$\begin{aligned} I = \int_C f(z)\,dz &= \int_C \frac{1}{z^2} f\left(\frac{1}{z}\right)\,dz \\ &= \int_C \frac{1}{z^2} e^z\,dz \\ &= \boxed{2\pi i} \end{aligned}$$

A common category of problems is that which contains the following two examples:

Example 33. Evaluate the integral,

$$I = \int_{-\infty}^{\infty} \frac{dx}{x^4 + 1}.$$

Solution. Let $f(z) = \dfrac{1}{z^4 + 1}$. We first note that the poles of f are ζ, ζ^3, ζ^5, and ζ^7, where $\zeta = e^{\pi i/4}$ is a principal eighth root of unity. Next we will employ a trick. Let C_R be the positively oriented semi-circle in the upper half plane. Then $\Gamma_R = C_R \cup [-R, R]$ form a positively oriented cycle which, when R is sufficiently large, bounds ζ and ζ^3. See Figure 10.3. Therefore, by the residue

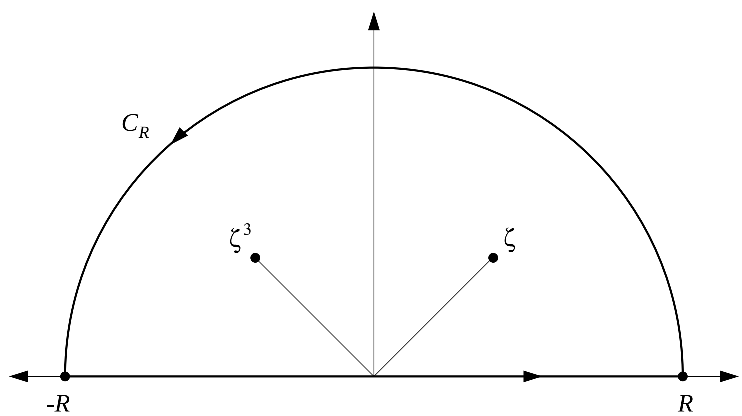

Figure 10.3: The cycle $\Gamma_R = C_R \cup [-R, R]$ bounds ζ and ζ^3.

theorem,

$$\int_{\Gamma_R} f(z)\, dz = \int_{-R}^{R} f(x)\, dx + \int_{C_R} f(z)\, dz = 2\pi i \left(\mathrm{Res}(f, \zeta) + \mathrm{Res}(f, \zeta^3) \right).$$

Let $g(z) = z^4 + 1$. Then

$$\mathrm{Res}(f, \zeta) = \frac{1}{g'(\zeta)},$$

and

$$\mathrm{Res}(f, \zeta^3) = \frac{1}{g'(\zeta^3)}.$$

A simple calculation gives $g'(\zeta) = 4i\zeta$, and $g'(\zeta^3) = -4i\zeta^3$. Thus,

$$\int_{\Gamma_R} f(z)\, dz = \frac{\pi}{\sqrt{2}}.$$

Now we claim that the line integral along C_R goes to zero as $R \to \infty$. This is easy to see, since $f(z) \to 0$ as $|z| \to \infty$ in the upper half plane. Therefore we conclude,

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = \boxed{\frac{\pi}{\sqrt{2}}}$$

□

Example 34. Evaluate the integral,

$$I = \int_0^{\infty} \frac{dx}{1+x^3}.$$

Solution. For this one, we will use a similar trick. We let $f(z) = \dfrac{1}{x^3+1}$ and note that the poles of f are at ζ, ζ^3, and ζ^5, where $\zeta = e^{\pi i/3}$ is a principal sixth root of unity. We will integrate around a cycle Γ_R, similar to that in Example 33. See Figure 10.4. Note that Γ_R is composed of C_R, the line segment $[0, R]$ and

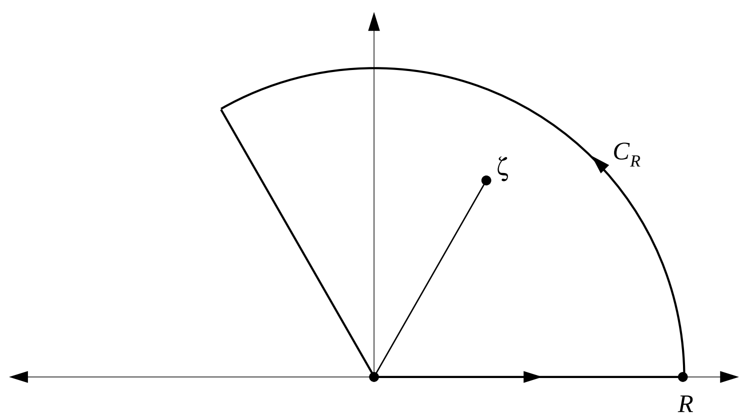

Figure 10.4: The cycle Γ_R bounds ζ.

the line segment of length R along $\theta = \frac{2\pi}{3}$, which we will call ℓ_R. Thus, by the residue theorem,

$$\begin{aligned}
\int_{\Gamma_R} f(z)\,dz &= \int_0^R f(x)\,dx + \int_{\ell_R} f(z)\,dz + \int_{C_R} f(z)\,dz. \\
&= 2\pi i \operatorname{Res}(f, \zeta).
\end{aligned}$$

We then note that we can make the substitution $z = xe^{2\pi i/3}$ into the integral along ℓ_R. Thus we obtain

$$\int_{\ell_R} f(z)\,dz = e^{2\pi i/3} \int_0^R f(x)\,dx.$$

Again, we have that as $R \to \infty$, the integral along C_R goes to zero. Thus, we have

$$(1-\zeta^2) \int_0^R \frac{dx}{x^3+1} = 2\pi i \operatorname{Res}(f, \zeta).$$

A quick calculation yields

$$\operatorname{Res}(f, \zeta) = \frac{1}{3\zeta^2}.$$

Therefore,

$$\int_0^\infty \frac{dx}{1+x^3} = \boxed{\frac{2\pi}{3\sqrt{3}}}$$

$\square$

Example 35. Let Γ be a closed path in $\mathbb{C} \setminus [0,1]$. Show that

$$\int_\Gamma \frac{1}{z(1-z)}\, dz = 0.$$

Proof. Let $f(z) = z(1-z)$. Then the only poles of $\dfrac{1}{f}$ are 0 and 1. Furthermore,

$$\operatorname{Ind}_\Gamma(0) = \operatorname{Ind}_\Gamma(1) = \pm k$$

depending on orientation. A quick calculation yields

$$\operatorname{Res}\left(\frac{1}{f}, 0\right) = \frac{1}{f'(0)} = 1,$$

and

$$\operatorname{Res}\left(\frac{1}{f}, 1\right) = \frac{1}{f'(1)} = -1.$$

Therefore,

$$\int_\Gamma \frac{1}{z(1-z)}\, dz = \pm k \cdot 2\pi i\,(1 + (-1)) = 0.$$

$\square$

Chapter 11

Harmonic Functions

The Cauchy-Riemann Equations

Definition (The Operators ∂ and $\overline{\partial}$). Define the differential operators ∂ and $\overline{\partial}$ by

$$\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \overline{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Theorem. Suppose f is a complex function in Ω that has a differential at every point in Ω. Then $f \in \mathscr{H}(\Omega)$ if and only if the Cauchy-Riemann equation

$$(\overline{\partial} f)(z) \equiv 0$$

holds for every $z \in \Omega$. In this case, we have

$$f'(z) = (\partial f)(z).$$

If $f = u + iv$ where u, v are real-valued functions, then the Cauchy-Riemann equation is

$$u_x = v_y, \quad u_y = -v_x.$$

Example 36. Let f be a complex valued function on Ω. Suppose that as a mapping of $\mathbb{R}^2$ to $\mathbb{R}^2$, $f = u + iv$ is twice differentiable, and that the Jacobian

$$J = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

is everywhere, as a linear transformation, a composition of a dilation and a rotation. Prove that $f \in \mathscr{H}(\Omega)$.

Proof. Since, at each $z \in \Omega$, J is the composition of a dilation and a rotation, there are real numbers $a > 0$ and $\theta \in [0, 2\pi)$, such that

$$J = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Therefore, $\frac{\partial u}{\partial x} = a\cos\theta = \frac{\partial v}{\partial y}$, and $\frac{\partial u}{\partial y} = a\sin\theta = -\frac{\partial v}{\partial x}$. The Cauchy-Riemann equations hold for f, and thus f is holomorphic. $\square$

Definition (Laplacian, Harmonic). Let f be a complex function in a plane open set Ω such that f_{xx} and f_{yy} exist at every point in Ω. Then we define the Laplacian of f to be

$$\Delta f = f_{xx} + f_{yy}.$$

If Δf is continuous on Ω and $\Delta f \equiv 0$ on Ω, then f is said to be harmonic in Ω.

Remark. Since the Laplacian of a real-valued function is real, it is clear that a complex function is harmonic if and only if both its real and imaginary parts are harmonic. Furthermore, if f has continuous second-order derivatives, then $f_{xy} = f_{yx}$ and $\Delta f = 4\partial\overline{\partial} f$.

Theorem. Holomorphic functions are harmonic.

Corollary (A partial converse). If f is harmonic, then ∂f is holomorphic.

Theorem. Every real harmonic function is locally the real part of a holomorphic function.

The Mean Value Property

Definition. We say that a continuous function u in an open set Ω has the mean value property if $u(z)$ is equal to the mean value of u on any circle centered at z. In other words, if C is a circle in Ω whose center is at z, then

$$u(z) = \frac{1}{2\pi}\int_C u(\theta)\, d\theta.$$

Theorem. If a continuous function u has the mean value property in an open set Ω, then u is harmonic in Ω.

Theorem (The Schwarz Reflection Principle). Suppose L is a segment of $\mathbb{R}$, Ω^+ a region in the upper half plane, and every $t \in L$ is the center of a disc, D_t whose upper half is contained in Ω^+. Let Ω^- be the reflection of Ω^+. Suppose $f = u + iv$ is holomorphic in Ω^+, and

$$\lim_{n\to\infty} v(z_n) = 0$$

if $z_n \to L$ (converges to a point in L).

Then there is an extention F which is holomorphic in $\Omega = \Omega^+ \cup L \cup \Omega^-$, such that $F\big|_{\Omega^+} = f$. Furthermore, $F(\overline{z}) = \overline{F(z)}$ for all $z \in \Omega$.

Chapter 12

The Maximum Modulus Principle

Definition. Define $\mathscr{H}^\infty$ to be the space of all bounded holomorphic functions in the unit disc U. The norm

$$\|f\|_\infty = \sup_{z \in U} |f(z)|$$

makes $\mathscr{H}^\infty$ Banach.

Theorem (The Schwarz Lemma). Suppose $f \in \mathscr{H}^\infty$, $\|f\|_\infty \leq 1$, and $f(0) = 0$. Then

$$|f(z)| \leq |z|, \quad \forall z \in U,$$
$$|f'(0)| \leq 1;$$

if equality holds in the first inequality somewhere away from 0, or if equality holds in the second inequality, then $f(z) = \lambda z$ where λ is a constant with $|\lambda| = 1$.

Example 37. Suppose that f and g are injective holomorphic functions of the open disk $U = \{z \in \mathbb{C} : |z| < 1\}$ onto some domain Ω. Suppose that $f(0) = g(0)$, and $f'(0) = g'(0) \neq 0$. Show that $f \equiv g$.

Proof. Since f is a bijection, f^{-1} exists and, by the inverse function theorem, is differentiable at all y such that $f(x) = y$ and $f'(x) \neq 0$.

Let $h = f^{-1} \circ g : U \longrightarrow U$. Then $h \in \mathscr{H}^\infty$ and $h(0) = 0$. Furthermore, by the chain rule,

$$\begin{aligned}
h'(0) &= \left(f^{-1}\right)'(g(0)) \cdot g'(0) \\
&= \left(f^{-1}\right)'(f(0)) \cdot f'(0) \\
&= \frac{1}{f'(0)} \cdot f'(0) \\
&= 1.
\end{aligned}$$

Therefore, by Schwarz's lemma, $h = \lambda z$, where $|\lambda| = 1$. Apply f to both sides of the equation $(f^{-1} \circ g)(z) = \lambda z$ to get $g(z) = f(\lambda z)$. Differentiate to get $g'(z) = \lambda f'(\lambda z)$. But since $g'(0) = f'(0) \neq 0$, we may conclude that $\lambda = 1$. Thus, h is the identity function and $f \equiv g$. $\square$

Definition. For any $\alpha \in U$ define

$$\varphi_\alpha(z) = \frac{z - \alpha}{1 - \overline{\alpha} z}.$$

Theorem. Fix $\alpha \in U$. Then φ_α is a bijection $U \longleftrightarrow U$ whose restriction to T, the unit circle, is a bijection $T \longleftrightarrow T$, and whose inverse is $\varphi_{-\alpha}$. Furthermore,

$$\varphi_\alpha(\alpha) = 0, \quad \varphi_\alpha(0) = -\alpha,$$

and

$$\varphi'_\alpha(0) = 1 - |\alpha|^2, \quad \varphi'_\alpha(\alpha) = \frac{1}{1 - |\alpha|^2}.$$

Example 38. Let $f : U \longrightarrow U$ be holomorphic. We will show that

$$|f(z) - f(0)| \leq \left| z \left(1 - \overline{f(0)} f(z) \right) \right|$$

for all $z \in U$.

Proof. Let $\alpha = f(0)$. Define φ_α as above, and let $g = \varphi_\alpha \circ f$. Then $g : U \to U$ is holomorphic, and $g(0) = 0$. Since φ_α is bounded, then g is bounded. Thus by the Schwarz Lemma,

$$|g(z)| \leq |z|$$

for all $z \in U$. Therefore,

$$\left| \frac{f(z) - f(0)}{1 - \overline{f(0)} f(z)} \right| \leq |z|.$$

The result follows. $\square$

Theorem. Suppose $f \in \mathscr{H}(U)$, f is injective, $f(U) = U$, and $f(\alpha) = 0$ for some $\alpha \in U$. Then there is a constant, λ, with $|\lambda| = 1$, such that

$$f(z) = \lambda \varphi_\alpha(z), \quad \forall z \in U.$$

Example 39. Let f be an automorphism of the unit disk, U. Assume that f fixes two points $\alpha, \beta \in U$. Show that $f(z) = z$ for all $z \in U$.

Proof. Consider φ_α as defined above. Then $\varphi_\alpha(\alpha) = 0$. Let $g = \varphi_\alpha \circ f$. Then $g : U \longrightarrow U$ is an automorphism and $g(\alpha) = 0$. Thus there exists a λ with $|\lambda| = 1$ such that $g(z) = \lambda \varphi_\alpha(z)$. But

$$\varphi_\alpha(\beta) = (\varphi_\alpha \circ f)(\beta) = g(\beta) = \lambda \varphi_\alpha(\beta).$$

Since $\varphi_\alpha(\beta) \neq 0$, we conclude that $\lambda = 1$. Thus, $\varphi_\alpha \circ f = \varphi_\alpha$ implies that f is the identity function. $\square$

Chapter 13

Approximation by Rational Functions

Theorem (Cauchy Formula). If K is a compact subset of a plane open set Ω, then there is a cycle $\Gamma \subseteq \Omega \setminus K$ such that the Cauchy Formula

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\xi)}{\xi - z}\, d\xi$$

holds for every $f \in \mathscr{H}(\Omega)$ and for every $z \in K$.

Theorem (Runge's Theorem). Let Ω be an open set in the plane, let A be a set consisting of one point in each component of $S^2 \setminus \Omega$, and suppose $f \in \mathscr{H}(\Omega)$. Then there is a sequence $\{R_n\}$ of rational functions, with poles only in A, such that $R_n \to f$ uniformly on compact subsets of Ω.

In the special case where $S^2 \setminus \Omega$ is connected, we may take $A = \{\infty\}$, and thus obtain polynomials $\{P_n\}$ with $P_n \to f$ uniformly on compact subsets of Ω.

Theorem (Mittag-Leffler Theorem). Suppose Ω is a plane open set, $A \subseteq \Omega$, A has no limit point in Ω, and to each $p \in A$ there are associated positive integers, $m(p)$ (the orders of the poles), and a rational function

$$Q_p(z) = \sum_{k=1}^{m(p)} \frac{c_{k,p}}{(z-p)^k}.$$

Then there exists a meromorphic function f in Ω, whose principal part at each $p \in A$ is Q_p, and which has no other poles in Ω.

Essentially, this says that we can construct a meromorphic function f with arbitrary preassigned poles, including orders.

Simply Connected Regions

Theorem. For a plane region Ω, the following (nine) are equivalent:

1. Ω is homeomorphic to the open unit disc U.
2. Ω is simply connected.
3. $\mathrm{Ind}_\gamma(\alpha) = 0$ for every closed path γ in Ω and for every $\alpha \in S^2 \setminus \Omega$.
4. $S^2 \setminus \Omega$ is connected.
5. Every $f \in \mathscr{H}(\Omega)$ can be approximated by polynomials, uniformly on compact subsets of Ω.
6. For every $f \in \mathscr{H}(\Omega)$ and every closed path γ in Ω,
$$\int_\gamma f(z)\,dz = 0.$$
7. To every $f \in \mathscr{H}(\Omega)$ corresponds an $F \in \mathscr{H}(\Omega)$ such that $F' = f$.
8. If $f \in \mathscr{H}(\Omega)$ and $\dfrac{1}{f} \in \mathscr{H}(\Omega)$, then there exists a $g \in \mathscr{H}(\Omega)$ such that $f = \exp(g)$.
9. If $f \in \mathscr{H}(\Omega)$ and $\dfrac{1}{f} \in \mathscr{H}(\Omega)$, then there exists a $\varphi \in \mathscr{H}(\Omega)$ such that $f = \varphi^2$.

Corollary. If $f \in \mathscr{H}(\Omega)$, where Ω is any plane open set, and f has no zeros in Ω then $\log|f|$ is harmonic in Ω.

Chapter 14

Conformal Mapping

Definition. A function f which is analytic at z_0 such that $f'(z_0) \neq 0$ is said to be conformal.

Conformal maps are those which preserve angles.

Linear Fractional Transformations

Definition. If a, b, c, and d are complex numbers such that $ad - bc \neq 0$, the mapping

$$z \mapsto \frac{az + b}{cz + d}$$

is called a linear fractional transformation.

It is convenient to regard this as a mapping from S^2 into itself. It is then easy to see that every linear fractional transformation is in fact a bijection $S^2 \longrightarrow S^2$. Furthermore, it is obtained by superposition of transformations of the following types:

1. Translations: $z \mapsto z + b$.
2. Rotations: $z \mapsto az, \quad |a| = 1$.
3. Dilations: $z \mapsto rz, \quad r > 0$.
4. Inversion: $z \mapsto \frac{1}{z}$.

Theorem (The Riemann Mapping Theorem). Every simply connected region in the plane (other than the entire plane itself) is conformally equivalent to the unit disk U.

Chapter 15

Zeros of Holomorphic Functions

Theorem (Weierstrass Factorization Theorem). Let Ω be a proper open set in S^2. Suppose $A \subseteq \Omega$ and A has no limit point in Ω. With each $a \in A$ associate a positive integer $m(a)$. Then there exists an $f \in \mathscr{H}(\Omega)$ all of whose zeros are in A, and such that f has a zero of degree $m(a)$ at each $a \in A$.

Theorem. Every meromorphic function in an open set Ω is the quotient of two functions which are holomorphic in Ω.

Algebraically, we are saying that $\mathscr{H}(\Omega)$ is a ring whose field of fractions is $\mathscr{M}(\Omega)$, the set of meromorphic functions on Ω.

An Interpolation Problem

Theorem. Suppose Ω is an open set in the plane, $A \subseteq \Omega$ with no limit point in Ω. To each $a \in A$ there is prescribed an integer $m(a)$ and complex numbers $w_{n,a}$ for $0 \leq n \leq m(a)$. Then there exists an $f \in \mathscr{H}(\Omega)$ such that

$$f^{(n)}(a) = n!\, w_{n,a},$$

for all $a \in A$.

This means that we can prescribe the values for f at each point $a \in A$, as well as finitely many derivatives.

Definition. Let $g_1, \ldots, g_n \in \mathscr{H}(\Omega)$. Then the ideal $[g_1, \ldots, g_n]$ is the set of all functions of the form $\sum f_i g_i$, where $f_i \in \mathscr{H}(\Omega)$. A principal ideal is one that is generated by a single function g. Note that $[1] = \mathscr{H}(\Omega)$.

Theorem. Every finitely generated ideal in $\mathscr{H}(\Omega)$ is principal.

Algebraists recognize this as "$\mathscr{H}(\Omega)$ is a PID" (thus a UFD).

Chapter 16

Analytic Continuation

Definition. A function element is an ordered pair (f, D), where D is an open circular disk and $f \in \mathscr{H}(D)$. Two function elements (f_1, D_1) and (f_2, D_2) are direct continuations of one another if two conditions hold: $D_1 \cap D_2 \neq \varnothing$, and $f_1(z) = f_2(z)$ for all $z \in D_1 \cap D_2$. In this case, we write

$$(f_1, D) \sim (f_2, D).$$

A chain is a finite sequence $\mathscr{C}$ of discs, $\mathscr{C} = \{D_0, D_1, \ldots, D_n\}$ such that $D_{i-1} \cap D_i$ is non-empty. If (f_0, D_0) is given and if there exists elements (f_i, D_i) so that $(f_{i-1}, D_{i-1}) \sim (f_i, D_i)$, then (f_n, D_n) is said to be the analytic continuation of (f_0, D_0) along $\mathscr{C}$.

Theorem. Analytic continuations along a curve, when they exist, are unique.

Theorem (Monodromy). Suppose Ω is a simply connected region, (f, D) is a function element, $D \subseteq \Omega$, and (f, D) can be analytically continued along every curve in Ω that starts at the center of D. Then there exists a $g \in \mathscr{H}(\Omega)$ such that $g\big|_D = f$.

Definition. The modular group, G, is the set of all linear fractional transformations φ of the form

$$\varphi(z) = \frac{az + b}{cz + d},$$

where $a, b, c, d \in \mathbb{Z}$, and $ad - bc = 1$ (as matrices, they have determinant 1). G is generated by the transformations $t_1(z) = z + 1$, and $-\mathrm{inv}(z) = -\frac{1}{z}$.

Theorem (Picard's Little Theorem). If f is entire and the image of f misses two distinct values, then f is constant.

Example 40. Let f be an entire function such that $f(\mathbb{C}) \subseteq \mathbb{C} \setminus S^1$, where S^1 is the unit circle. Show that f is a constant.

Proof. f misses 1 and i. Thus, since it misses two points, f is constant. $\square$

Example 41. Let $W = \left\{z = x + iy \in \mathbb{C} \,:\, y^2 - x^2 < 1\right\}$ be the region bounded by a hyperbola. Let f be an entire function such that $f(\mathbb{C}) \subseteq W$. Show that f is a constant function.

Proof. There are many points outside of W. In particular, $\pm 2i \notin W$. Thus, since f misses two points, it must be a constant. $\square$

Example 42. Let f be an entire function such that $|\Re f(z)| + |\Im f(z)| \geq 1$ for each $z \in \mathbb{C}$. Prove that f is a constant function.

Proof. There are many values missed by f: 0 and $\frac{1}{2}$ are two examples. Therefore, by Picard's little theorem, f is constant. $\square$

Example 43. Let f be a nonconstant entire function. Show that for every constant $k \in \mathbb{R}$, there exists a $z \in \mathbb{C}$ such that $\Im f(z) = k \cdot \Re f(z)$.

Proof. Let $k \in \mathbb{R}$. Consider the line in $\mathbb{C}$ given by

$$\ell = \{t + kt\,i \,:\, t \in \mathbb{R}\}.$$

Since f is entire and nonconstant, $\mathbb{C} \setminus f(\mathbb{C})$ consists of at most one point. Therefore, there is at least some $t_0 \in \mathbb{R}$ so that $t_0 + kt_0 i \in f(\mathbb{C})$. Thus, $t_0 + kt_0 i = f(z_0)$ for some $z_0 \in \mathbb{C}$. This is the point we seek. $\square$

Theorem (Picard's Theorem). Every entire function which is not a polynomial attains each value infinitely many times, with one possible exception.

To see that the exceptional value may be taken a nonzero finite number of times, let $p(z)$ be a polynomial of degree n with only simple zeros. Then $f(z) = p(z)e^z$ attains the value zero exactly n times.

Example 44. Let f be an injective entire function. Put $g(z) = f(1/z)$. Show that g does not have an essential singularity at 0. In particular, it follows that $f(z) = az + b$.

Proof. By Picard's theorem, if f is injective and entire, then f is polynomial. First we note that f cannot be constant, since it is injective. Second, if f is polynomial and $\deg f = n \geq 2$, then we will arrive at a contradiction, as follows.

By the Fundamental Theorem of Algebra, f can be factored uniquely as a product of irreducible linear factors. If f has two or more distinct linear factors, say

$$(z - \alpha), \quad \text{and} \quad (z - \beta),$$

where $\alpha \neq \beta$, then f is not injective on $f^{-1}(\{0\})$.

On the other hand, if f has only a repeated factor, then

$$f(z) = a(z - \alpha)^n.$$

Now choose z so that $z - \alpha$ is an nth root of unity (there are n such values for z). Then, since $n \geq 2$, f is not injective on $f^{-1}(\{a\})$. We conclude that f must be a polynomial of degree one. Hence f is linear, and g has a simple pole at 0. □

Example 45. Suppose that f is meromorphic on the plane, is never zero on the plane, and that

$$\lim_{z \to \infty} |f(z)| = 0.$$

Show that f is the inverse of a polynomial.

Proof. Let A be the set of poles of f. If A is infinite then we claim that A has a limit point at ∞. Indeed, S^2 is compact, so by Bolzano-Weierstrass, A must have a limit point somewhere on the sphere. However, by definition of meromorphic, A cannot have a limit point in $S^2 \setminus \{\infty\}$. But if ∞ is a limit point for the set of poles, we arrive at a contradiction since

$$\lim_{z \to \infty} |f(z)| = 0.$$

Therefore, we conclude that A is finite, and $\dfrac{1}{f}$ has only finitely many zeros in the plane, and has a singularity at ∞. Therefore, $\dfrac{1}{f}$ is a polynomial. □

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